MATS UNIVERSITY

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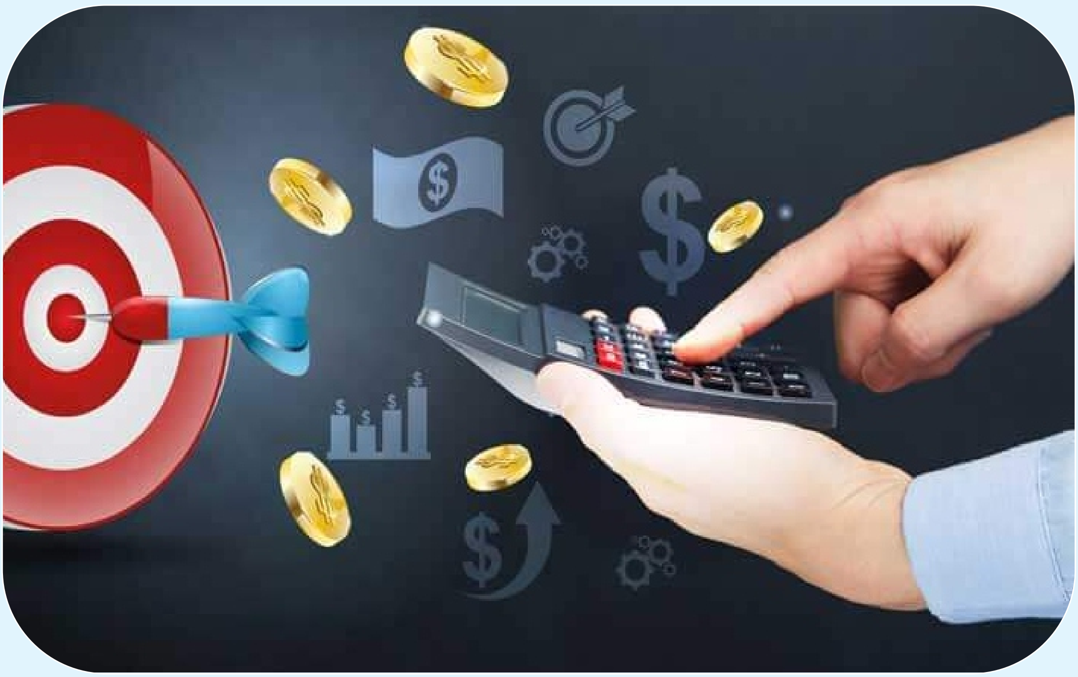
MATS CENTRE FOR OPEN & DISTANCE EDUCATION

Business Mathematics

Bachelor of Commerce (B.Com.)

Semester - 3



SELF LEARNING MATERIAL

ODL/BCOM DSC-009
Business Mathematics

Business Mathematics
CODE : ODL/BCOM DSC-009

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Module Introduction

Course has five Modules. Under this theme we have covered the following topics:

Module I AVERAGES, RATIO & PROPORTION, PERCENTAGE, AND DISCOUNT

Module II MATRICES AND DETERMINANTS

Module III SIMPLE INTEREST, COMPOUND INTEREST, AND PROFIT & LOSS

Module IV LINEAR PROGRAMMING AND TRANSPORTATION PROBLEM

Module V THEORY OF INDICES AND LOGARITHMS

These themes are dealt with through the introduction of students to the foundational concepts and practices of effective management. The structure of the MODULES includes these skills, along with practical questions and MCQs. The MCQs are designed to help you think about the topic of the particular MODULE. We suggest that you complete all the activities in the modules, even those that you find relatively easy. This will reinforce your earlier learning.

We hope you enjoy the MODULE.

If you have any problems or queries, please contact us:

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MODULE I AVERAGES, RATIO & PROPORTION, PERCENTAGE, AND DISCOUNT

Structure

OBJECTIVES

- Develop a fundamental understanding of averages and their applications in problem-solving.
- Explain ratio and proportion concepts with practical applications.
- Enhance calculation skills related to percentages, including profit, loss, and discount.
- Improve quantitative problem-solving abilities through real-world examples

Unit 1 AVERAGES

An average is a single value that represents the central or typical value of a dataset. The most common types of averages are:

1. **Arithmetic Mean**: The sum of all values divided by the number of values.

- Formula: Mean = $(x_1 + x_2 + ... + x_n)/n$

- Example: For the dataset {4, 7, 9, 12, 18}, the arithmetic mean is (4 + 7 + 9 + 12 + 18)/5 = 50/5 = 10

2. **Median**: The middle value when data is arranged in ascending or descending order.

- For odd number of values: The middle element
- For even number of values: Average of the two middle elements
- Example: For {4, 7, 9, 12, 18}, the median is 9

3. **Mode**: The value that appears most frequently in a dataset.

- Example: For {4, 7, 7, 9, 12, 18}, the mode is 7

4. **Range**: The difference between the maximum and minimum values.

- Example: For {4, 7, 9, 12, 18}, the range is 18 - 4 = 14

5. **Geometric Mean**: The nth root of the product of n values.

- Formula: GM = $\sqrt[n]{(x_1 \times x_2 \times ... \times x_n)}$
- Example: For {2, 4, 8}, the geometric mean is $\sqrt[3]{(2 \times 4 \times 8)}$ = $\sqrt[3]{64}$ = 4

6. **Harmonic Mean**: The reciprocal of the arithmetic mean of the reciprocals.

- Formula: HM = $n/(1/x_1 + 1/x_2 + ... + 1/x_n)$
- Example: For {2, 4, 8}, the harmonic mean is 3/(1/2 + 1/4 + 1/8) = 3/(7/8) = 24/7 ≈ 3.43

Properties and Applications

1. **Properties**:

- The arithmetic mean is influenced by extreme values (outliers)
- Mean ≥ Geometric Mean ≥ Harmonic Mean (for positive numbers)

- The sum of deviations from the mean is always zero
- For symmetric distributions, mean = median = mode

2. **Applications**:

- **Arithmetic Mean**: Used for calculating average scores, average income, average temperature
- **Median**: Used when data contains outliers (e.g., housing prices, income distribution)
- **Mode**: Used for categorical data (e.g., most common hair color, most popular product)
- **Geometric Mean**: Used for growth rates, investment returns, and averaging ratios
- **Harmonic Mean**: Used for averaging rates, speeds, and prices

Weighted Averages

A weighted average assigns different weights (importance) to different values.

Formula: Weighted Average = $(w_1x_1 + w_2x_2 + ... + w_nx_n)/(w_1 + w_2 + ... + w_n)$

Applications:

- GPA formula where the credit are the weights
- Investment portfolio returns (with investment amounts as weights)
- Teacher marks different assignments with different percentages
- Price indices where quantities are weights

Numerical Problems

Problem 1: Arithmetic Mean

The daily temperatures (in °C) for a week were: 24, 26, 28, 23, 25, 27, 29. What is the average temperature for the week?

Solution: Mean = (24 + 26 + 28 + 23 + 25 + 27 + 29)/7 = 182/7 = 26°C

Problem 2: Median

The median is, as you know, the middle value in a sorted order of values.

Solution: 8, 12, 15, 17, 19, 22, 31 Arrange in ascending order: 8, 12, 15, 17, 19, 22, 31 The number of numbers is 7 (odd) The fourth number is 17

Problem 3: Mixed Averages

The scores of a student in 5 tests are 78, 83, 90, 88, and 76. Find the: a) Arithmetic mean b) Median c) Range

Solution: a) Mean = (78 + 83 + 90 + 88 + 76)/5 = 415/5 = 83 b) Arranged scores: 76, 78, 83, 88, 90; Median = 83 c) Range = 90 - 76 = 14

Problem 4: Weighted Average

A student has the following marks in different subjects with different credits:

- Math: 85 (4 credits)
- Science: 78 (3 credits)
- English: 92 (3 credits)
- History: 88 (2 credits)

Calculate the weighted average (GPA).

Solution: Weighted Average = (85×4 + 78×3 + 92×3 + 88×2)/(4 + 3 + 3 + 2) = (340 + 234 + 276 + 176)/12 = 1026/12 = 85.5

Problem 5: Geometric Mean

Find the geometric mean of 4, 8, and 25.

Solution: The geometric mean (GM) of the numbers 4, 8, and 25 is calculated as follows:

$$GM - \sqrt[3]{4 \times 8 \times 25}$$

First, multiply the numbers:

$$4 \times 8 - 32$$

$$32 \times 25 - 800$$

Therefore,

$$GM- \sqrt[3]{800}$$

Taking the cube root of 800 gives:

$$GM \approx 9.28GM$$

Hence, the geometric mean of 4, 8, and 25 is approximately 9.289.

Problem 6: Mean from Frequency Distribution

Find the mean from the following frequency distribution:

Value	10	20	30	40	50
Frequency	3	7	12	8	5

Solution: The mean of the given data is calculated as follows:

The formula for the mean is:

$$\text{Mean} = \frac{\sum(x \times f)}{\sum f}$$

Given data:

$$(10\times3)+(20\times7)+(30\times12)+(40\times8)+(50\times5)$$

Summing the products:

$$30+140+360+320+250-110$$

Summing the frequencies:

$$3+7+12+8+5-35$$

Substituting in the formula:

$$\text{Mean} = \frac{1100}{35} \approx 31.43$$

Therefore, the mean of the given data is approximately **31.43**.

Problem 7: Harmonic Mean

Find the harmonic mean of 2, 4, and 8.

Solution: The formula for the harmonic mean (HM) of

n values is:

$$HM = \frac{n}{\sum \left(\frac{1}{x_i}\right)}$$

For 2, 4, and 8:

$$HM = \frac{3}{\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right)}$$

$$= \frac{3}{\left(\frac{4}{8} + \frac{2}{8} + \frac{1}{8}\right)}$$

$$= \frac{3}{\frac{7}{8}} = 3 \times \frac{8}{7} = \frac{24}{7} \approx 3.43$$

Problem 8: Effect of Adding a Value

If the mean of five numbers is 15, what will be the new mean if a sixth number, 21, is added to the dataset?

Solution:

Sum of the five numbers:

$$15 \times 5 = 75$$

New sum after adding 21:

$$75 + 21 = 96$$

New mean:

$$\frac{96}{6} = 16$$

Problem 9: Finding a Missing Value

The average of 6 numbers is 42. If one of the numbers is excluded, the average of the remaining numbers is 40. What is the excluded number?

Solution:

Total sum of six numbers:

$$42 \times 6 = 252$$

Total sum of the remaining five numbers:

$$40 \times 5 = 2004$$

Excluded number:

$$252 - 200 = 52$$

Problem 10: Average Speed

A car travels 120 km at 60 km/h and then another 120 km at 40 km/h. What is the average speed for the entire journey?

Solution: Total distance = 120 + 120 = 240 km Time for first part = 120/60 = 2 hours Time for second part = 120/40 = 3 hours Total time = 2 + 3 = 5 hours Average speed = Total distance/Total time = 240/5 = 48 km/h

Note: This is the harmonic mean of the two speeds weighted by the distances.

Unit 2RATIO & PROPORTION

It indicates how many times one value contains another value. Ratios are usually written in the form a:b and read as "a to b," or as a fraction a/b. However, proportion is an equation that two ratios are equal. When a:b = c:d, we call a, b, c, and d to be in proportion, and we denote it by writing a:b::c:d (read: a is to b as c is to d).

Key Properties of Ratios

1. **Simplification**: A ratio can be simplified, or reduced by dividing both numbers by their greatest common divisor (GCD). 13 a.m. for 8:12 to be reduced to 2:3.

2. **Equivalent Ratios**: A dilation transforms a figure, enlarging or reducing its size while keeping its shape. For example, 2:3 is the same as four to six, as they multiply out to equal the same number..

3. **Comparison**: Ratios can be compared by bringing the two ratios to the same denominator or converting them to decimals

4. **Composition**: If a:b = c:d, then (a+b):b = (c+d):d and a:(a+b) = c:(c+d).

Key Properties of Proportions

1. **Fundamental Property**: If a:b = c:d, then ad = bc. This is known as the "cross multiplication" property.

2. **Alternate Proportion**: If a:b = c:d, then a:c = b:d.

3. **Componendo and Dividendo**: If a:b = c:d, then (a+b):b = (c+d):d (componendo) and (a-b):b = (c-d):d (dividendo).

4. **Continued Proportion**: If a:b = b:c, then b is the mean proportional between a and c, and $b^2 = ac$.

Direct and Inverse Proportions

Direct Proportion: Two quantities are directly proportional if when one increases, the other increases in direct relation, or when one decreases, the other decreases in direct relation.

Essentially, mathematically speaking, if x and y are directly proportional, we write: $y \propto x$ or $y = kx$, where k is a constant of proportionality.

Examples of Direct Proportion:

1. If a car moves with a constant speed, the distance travelled by it is directly proportional to the time taken.
2. Identical items will cost the same amount for each, and the total cost of the items purchased is proportional to the number purchased.
3. The weight of an object depends on its mass W = mg; where g is the acceleration due to gravity.

Inverse Proportion: When the items are inversely related, one item gets bigger, which makes the other smaller proportionately.In mathematical lingo, ᶦf x and y are inversely proportional, we write: y ∝ 1/x or y = k/x, where k is ι constant of proportionality.

Examples of Inverse Proportion:

- It takes less time to complete the task when more workers are working on it.
- Answer: The pressure of a gas is inversely related to its volume at constant temperature (Boyle's Law).
- Light intensity is inversely proportional to the distance squared from the source

Practical Applications

Architecture and Art

- **Golden Ratio**: The golden ratio (⅓ = ~ 1.618:1) is present in structures, works of art, and nature itself. It's thought to be aesthetically attractive and has played its part in structures like the Parthenon and works like the Mona Lisa..
- **Scale Models**: Creating Blue Prints and Models of Buildings 1:50 scale, for example, means that 1 unit on the model equals 50 units in the real structure..

Cooking and Baking

- **Recipe Scaling**: Many recipes use ratios so they can be scaled up or down. A simple bread recipe, for instance, may call for a ratio of 5:3 of flour to water.
- **Ingredient Proportions**: Flavors and textures are determined by the ratio of the ingredients in a recipe. The ratio of fat to flour, for example, determines the flakiness of pastry,

Finance and Economics

- **Financial Ratios**: Businesses use ratios to analyze the financial health of a company. These include the debt-to-equity ratio, profit margin ratio, and current ratio.

- **Interest Rates**: simple and compound interest. For instance, when interest earned is directly proportional to the principal a/c and the rate of interest

Engineering and Physics

- **Gears and Pulleys**: The number of teeth on two gears relative to the distance of movement of the elements. The mechanical advantage, in the same manner, is governed by the ratio of the radii of pulleys.

- **Stress and Strain**: Within the elastic limit (Hooke's law), stress is directly proportional to strain in materials science..

- **Electrical Circuits**: Voltage Drop in Series Circuits.

Medicine and Health

- **Drug Dosages**: The dosage of drugs is typically computed in relation to body weight or body surface area, allowing us to achieve a ratio between the dose and the dimensions of the patient.

- **BMI**: It is the weight divided by the square of the height..

Transportation and Navigation

- **Map Scales**: Maps show distances using ratios. A 1:10,000 scale, for example, indicates that 1 cm on the map is analogous to 10,000 cm (100 m) in actual life.
- **Fuel Consumption**: Fuel efficiency is often expressed as a ratio of distance traveled to fuel consumed, such as miles per gallon (mpg) or kilometers/liter (k/l).

Population Studies

- **Sex Ratio**: The ratio between the male and female of a population.

- **Dependency Ratio**: Young and elderly dependents compared to working-age population.

Chemistry

- **Chemical Equations**: In a balanced chemical equation, the coefficients represent the ratio of moles of reactants to moles of products.
- **Concentration**: Concentration is the ratio of solute to solvent or solute to solution

Examples and Numerical Problems

Basic Ratio Problems

Example 1: Simplifying Ratios Simplify the ratio 24:36.

Solution: To simplify a ratio, we divide both terms by their greatest common divisor (GCD).

Given ratio: 24:36

Step 1: Find the GCD of 24 and 36
The factors of 24 are: 1, 2, 3, 4, 6, 8, 12, 24
The factors of 36 are: 1, 2, 3, 4, 6, 9, 12, 18, 36
The common factors are: 1, 2, 3, 4, 6, 12
The greatest common divisor (GCD) is 12.

Step 2: Divide both terms by the GCD

$$\frac{24}{12} - 2$$
$$\frac{36}{12} - 3$$

Step 3: Write the simplified ratio
Therefore, the simplified ratio is:

24:36-2 : 3

Hence, 24:36 in its simplest form is 2:3.

Example 2: Equivalent Ratios Find four equivalent ratios for 3:4.

Solution: Finding Equivalent Ratios:

1. Multiplying both terms by 2:

$$3\times2:4\times2\text{-}6:8$$

Multiplying both terms by 3:

$$3\times3:4\times3=9:12$$

Multiplying both terms by 4:

$$3\times4:4\times4=12:16$$

Multiplying both terms by 5:

$$3\times5:4\times5=15:20$$

Example 3: Comparing Ratioscompare the ratios 5:8 and 7:12.

Solution: Step 1: Convert Ratios to Fractions

The given ratios are **5:8** and **7:12**.

$$5:8-\frac{5}{8}-0.625$$

$$7:12-\frac{7}{12}-0.583$$

Step 2: Compare the Decimals

Since 0.625 > 0.583, we conclude that:

$$5:8>7:12$$

Example 4: Dividing a Quantity in a Given Ratio Divide 350 in the ratio 2:3:5.

Solution: Step 1: Find the Sum of the Ratio Parts

Add the parts of the ratio:

$$2+3+5=10$$

Step 2: Calculate the Value of Each Unit

Divide the total amount by the sum of the ratio parts:

$$\frac{350}{10} - 35$$

Each part of the ratio represents 35 units**.**

Step 3: Calculate Each Share

First part:

2×35=702

Second part:

3×35-105

Third part:

5×35- 1755

Example 5: Finding a Ratio from Given Quantities A box contains 12 red marbles and 18 blue marbles. Find the ratio of red marbles to blue marbles in its simplest form.

Solution: The ratio of red marbles to blue marbles is 12:18. To simplify, we find the GCD of 12 and 18, which is 6. 12 ÷ 6 = 2 18 ÷ 6 = 3 Therefore, the ratio of red marbles to blue marbles is 2:3 in its simplest form.

Proportion Problems

Example 6: Verifying a Proportion Determine whether the following pairs of ratios form a proportion: 4:5 and 16:20.

Solution: We can check this by cross-multiplying. 4 × 20 = 80 5 × 16 = 80 Because the cross-products are equal, the two ratios, 4:5 and 16:20 are proportional. We can express this as 4:5 = 16:20 or 4:5::16:20.

Example 7: Missing Term in a Proportion If 3:x = 12:20, what is the value of x?

Using the basic property of proportion, this can be solved: 3 × 20 = x × 12 60

Example 8: Mean Proportional The mean proportional of 8 and 18.

Solution: Assume mean proportional between 8 and 18 as x. Now, 8:x = x:18 By the basic property: 8 × 18 = x^2 144 = x^2 x = √144 = 12 Thus, the mean proportional between 8 and 18 is 12.

Example 9: The Third Proportional Find the third proportional to 6 and 9.

Solution: Finding the Third Proportional

To find the third proportional to 6 and 9**,** we assume the third proportional to be **x**.

Step 1: Set Up the Proportion

The key concept of the third proportional can be written as:

$$6 : 9 - 9 : x$$

Step 2: Apply the Proportion Property

Using the property of proportions:

$$6 \times x - 9 \times 9$$

$$6x - 81$$

Step 3: Solve for x

$$x - \frac{81}{6} - 13.5$$

Conclusion:

The third proportional to 6 and 9 **is** 13.5**.**

Direct Proportion Problems

Example 11: Distance Travelled A car covers 210 km in 3 hours at a uniform speed. If it maintains the same speed, how many miles will it go in 5 hours?

Solution: Step 1: Understand the Relationship

Since the car travels at a uniform speed**,** the distance traveled **is** directly proportional to the time spent traveling**.**

Step 2: Set Up the Proportion

Let the distance traveled in **5** hour**s** be **x** km.

Using the proportion property:

$$\frac{210}{x} = \frac{3}{5}$$

Step 3: Cross Multiply

$$210 \times 5 = x \times 3$$
$$1050 = 3x$$

Step 4: Solve for x

$$x = \frac{1050}{3} = 350$$

Example 12: How much will it cost to buy 15 books of the same kind if 8 books of the same kind cost $120?

Solution: Example: Calculating the Cost of Books

Let the cost of 15 books be x**.**

Since the cost is directly proportional to the number of books, we can set up the proportion as follows:

$$\frac{15}{n} = \frac{120}{x}$$

Step 1: Cross Multiply

$$15\times x=120\times n$$

Step 2: Substitute the Given Values

Given that n = 8**,** the equation becomes:

$$15 \times x - 120 \times 8$$
$$15x - 960$$

Step 3: Solve for xxx

$$x - \frac{960}{15} - 64$$

Conclusion:

The cost of 15 books **is** $64**.**

Example: Calculating Output of Workers

Let the output produced by 18 workers be x**.**
Since the output **is** directly proportional to the number of workers, we can write:

$$\frac{12}{18} - \frac{144}{x}$$

Step 1: Cross Multiply

$$12 \times x - 18 \times 144$$
$$12x - 2592$$

Step 2: Solve for xxx

$$x - \frac{2592}{12} - 216$$

Conclusion:

18 workers can produce **216 units** in **one day**.

Example 14: Worker Productivity 8 workers can complete a project in 20 days. How many days will it take to complete the project, if 4 additional workers of the same efficiency join the team?

Solution: Let **x** represent the number of days required to complete the project with **12 workers**.

Since the number of workers and the number of days are inversely proportional (more workers reduce the number of days), we can use the principle of joint variation:

$$8 : 12 = x : 20$$

Applying the principle of joint variation:

$$8 \times 20 = 12 \times x$$
$$160 = 12x$$
$$x = \frac{160}{12}$$
$$x = 13.33 \text{ days}$$

Example 15: Speed of a train A train in 5 hours covers a distance of 450 km. And how long would it take to cover a distance of 720 km at the same speed?

Answer: Let **x** represent the time taken to cover 720 km.

Since the speed is constant, the time taken is directly proportional to the distance covered. Therefore:

$$\frac{450}{720} = \frac{5}{x}$$

Applying the basic property of proportion:

$$450 \times x = 720 \times 5$$
$$450x = 3600$$
$$x = \frac{3600}{450}$$
$$x = 8$$

Inverse Proportion Problems

Data used for challenges must be clearly notating the source of the data, clearly showcasing from where you are using the data. Let x hours be the time taken at

90 km/hExplanation:It is given that the time taken at 90 km/h is x hours. Because distance is constant, the time taken is inversely proportional to the velocity: 6:x = 90:60 Using the basic property: 6 × 60 = x × 90 360 = 90x x = 360 ÷ 90 = 4 Therefore, It will consume 4 hours to cover a same distance of 90 km/h.

Example 17: About workers and time If 12 workers can complete a task in 15 days, how many days will it take for 20 workers to complete the same task, assuming all workers work at the same rate?

Solution : Let **x** represent the number of days required for **20 workers** to complete the task.

Since the total work remains constant, the number of days is **inversely proportional** to the number of workers. Therefore:

$$\frac{12}{20} = \frac{x}{15}$$

Applying the fundamental property of inverse proportion:

$$12 \times 15 = 20 \times x$$

$$180 = 20x$$

$$x = \frac{180}{20}$$

$$x = 9$$

Example 18: Filling a Tank in 20 Minutes How long does it take for 5 same taps to fill?

Solution: Let **x** be the time (in minutes) it takes for **5 taps** to fill the tank.

Since the amount of work (filling the tank) is constant, the time taken is inversely proportional to the number of taps. This gives us the proportion:

$$\frac{1}{5} = \frac{x}{20}$$

Using the basic property of inverse proportion, we have:

$$1 \times 20 = 5 \times x$$

This simplifies to:

$$20 = 5x$$

Solving for **x**:

$$x = \frac{20}{5} = 4$$

Example 19: Pressure and Volume The gas has a pressure of 12 atmospheres at a volume of 5 liters. What will be the new pressure if the volume is increased to 8 liters at the same temperature?

Answer Let the new pressure be **x** atmospheres.

According to Boyle's Law, at constant temperature the pressure of a gas is inversely proportional to its volume. This implies:

$$P_1 \times V_1 = P_2 \times V_2$$

Given:
$P_1 = 12$ atm, $V_1 = 5$ L, $V_2 = 8$ L, and $P_2 = x$ atm.

Substitute the values into the equation:

$$12 \times 5 = x \times 8$$

Simplify the equation:

$$60 = 8x$$

Now, solve for **x**:

$$x = \frac{60}{8} = 7.5$$

Ex1: Brightness of Light At 3 meters away from a point source, the brightness of the light is 100 candelas. What will the intensity be at 6m?

Solution: Let the intensity at **6** meters be **x** candelas.

According to the inverse square law, the brightness of a light is inversely proportional to the square of the distance from the source. Therefore:

$$\frac{100}{x} = \frac{6^2}{3^2}$$

Calculating the squares:

$$\frac{100}{x} = \frac{36}{9}$$

Simplify the fraction on the right:

$$\frac{100}{x} = 4$$

Now, solve for **x** by cross-multiplying:

$$100 = 4x$$

$$x = \frac{100}{4} = 25$$

Combined Proportion Problems

Example 21: Work and Time If 15 men can complete a wall within 12 days working 8 hours a day, then how long would it take 20 men to complete the same wall working 6 hours a day?

Solution: Let **x** be the number of days required for 20 men working 6 hours per day to complete the wall.

The total work done is directly proportional to:

- The number of workers
- The number of days
- The number of hours worked per day

For the first scenario, the total work (in man-hours) is:

$$\text{Work} = 15 \times 12 \times 8 = 1440 \text{ man-hours}$$

For the second scenario, the total work is:

$$\text{Work} = 20 \times x \times 6 = 120x \text{ man-hours}$$

Since the work is the same in both cases, set the expressions equal:

$$1440 = 120x$$

Solve for **x**:

$$x = \frac{1440}{120} = 12$$

Example 22: Fuel Consumption A vehicle consumes 30 litres of fuel for a 360 km journey at an average speed of 60 km/h. How much fuel will it consume for a 480 km journey at an average speed of 80 km/h?

Solution:

Let **x** be the fuel consumption (in liters) for the 480 km journey.

Given that fuel consumption is:

- Directly proportional to the distance traveled, and
- Inversely proportional to the speed (assuming constant engine efficiency),

We can set up the proportion as:

$$\frac{30}{x} = \frac{480 \times 60}{360 \times 80}$$

Step 1: Calculate the product for the new journey and the original journey:

$$480 \times 60 = 28800$$

$$360 \times 80 = 28800$$

Step 2: Substitute these values into the proportion:

$$\frac{30}{x} = \frac{28800}{28800}$$

Step 3: Solve for **x**:

$$x = 30 \text{ liters}$$

Example 23: Wall Painting If 8 painters can paint an area of a wall 400 m² in 6 days, how many painters would be required to paint the wall area of 600 m² in 4 days?

Solution: Solution:

Let the required number of painters be xxx.

Concept:

- The number of painters is directly proportional to the area of the wall to be painted.
- The number of painters is inversely proportional to the number of days available for painting.

Setting Up the Proportion:

$$\frac{8}{x} = \frac{600 \times 6}{400 \times 4}$$

Simplifying the Ratios:

$$\frac{8}{x} = \frac{3600}{1600}$$
$$\frac{8}{x} = \frac{9}{4}$$

Cross-Multiplying to Solve for xxx:

$$8 \times 4 = x \times 9$$
$$32 = 9x$$
$$x = \frac{32}{9}$$
$$x = 3.56$$

Rounding to the nearest integer, the required number of painters is **4**.

Example 24: Irrigation. If it is to irrigate 21 hectares of land, working 6 hours a day − on how many days they will take to irrigate that land?

Solution: Let the number of days required be xxx.

Concept:

- The number of days is directly proportional to the area of land to be irrigated.
- The number of days is inversely proportional to the number of working hours per day.

Setting Up the Proportion:

$$\frac{8}{x} = \frac{21 \times 7}{12 \times 6}$$

Simplifying the Ratios:

$$\frac{8}{x} = \frac{147}{72}$$

Cross-Multiplying to Solve for x:

$$8 \times 72 = x \times 147$$

$$576 = 147x$$

$$x = \frac{576}{147}$$

$$x = 3.92$$

Rounding to the nearest integer, the number of days required is 4.

Answer:

It would take approximately 4 days to irrigate 21 hectares of land.

Example 25: How to Feed Cattle A farmer has sufficient food to feed 20 cows for 15 days. How long will the food be sufficient for the remaining cows if he sells 5 cows?

Solution:

Step 1: Understanding the Concept

The number of days the food will last is inversely proportional to the number of cows.

Step 2: Calculating Total Food Supply (Cow-Days)

Since 20 cows can be fed for 15 days, the total amount of food available is:

$$\text{Total Food} = 20 \times 15 = 300 \text{ cow-days}$$

Step 3: Calculating the New Number of Cows

After selling 5 cows, the remaining number of cows is:

$$20 - 5 = 15 \text{ cows}$$

Step 4: Calculating the Number of Days the Food Will Last

Since the total food supply is 300 cow-days, and there are now 15 cows to feed, the number of days the food will last is:

$$\text{Number of Days} = \frac{300}{15} = 20 \text{ days}$$

Solving some tough Ratio & Proportion Problems

Example 26: Compound Interest If an investment of $1000 at q% compound interest yields $1210 in a period of 2 years, how much will yield after 3 years?

Solution:

Step 1: Understanding the Compound Interest Formula

The compound interest formula is given by:

$$A = P(1 + r)^t$$

Where:

- AAA = Final amount
- P = Principal amount
- r = Annual interest rate (as a decimal)
- t = Time in years

Step 2: Identifying Given Values

- P = $1000 (Initial Principal)
- r = 10% = 0.10 (Annual Interest Rate)
- t = 3 years

Step 3: Substituting Values into the Formula

$$x = 1000 \times (1 + 0.1)^3$$

Step 4: Calculating the Amount

$$x = 1000 \times (1.1)^3$$
$$x = 1000 \times 1.331$$
$$x = 1331$$

Answer:

The amount after 3 years will be 1331 TL.

Example 27: Mixture and Alligation A vessel contains 60 lts of mixture of milk and water in the ratio of 2:3. How much milkneeds to be added to achieve a ratio of 3:2?

Step 1: Identify the Initial Quantities

Step 1: Identify the Initial Quantities

Let the amount of milk be m and the amount of water be www.

Since the ratio of milk to water is **2:3**, we can write:

$$\frac{m}{w} = \frac{2}{3}$$

The total volume of the mixture is 60 liters**:**

$$m + w = 60$$

Using the ratio to find the amount of milk:

$$m = \frac{2}{5} \times 60 = 24 \text{ liters}$$

And the amount of water:

$$w = \frac{3}{5} \times 60 = 36 \text{ liters}$$

Step 2: Adding Milk to Achieve the New Ratio

To change the ratio to 3:2**,** let xxx be the liters of milk to be added.

The new ratio equation becomes:

$$\frac{24 + x}{36} = \frac{3}{2}$$

Cross-Multiplying:

$$(24 + x) \times 2 = 36 \times 3$$

$$48 + 2x = 108$$

Solving for xxx:

$$2x = 108 - 48$$

$$2x = 60$$

$$x = \frac{60}{2}$$

$$x = 30$$

Final Answer:

To achieve a **3**:2 ratio**,** 30 liters of milk must be added.

Example 28: Mixture of Different Concentrations A solution has 20% alcohol. A second solution is 50% alcohol. What proportion of these solutions should be mixed to obtain a solution with 30% alcohol?

Let the ratio of the first solution to the second solution be **x : y**.

Using the rule of allegation:

$$(x \times 20\%) + (y \times 50\%) - (x + y) \times 30\%$$

Substituting values:

$$20x + 50y - 30x + 30y$$

$$20y - 10x$$

$$y - \frac{x}{2}$$

Therefore, the ratio of the first solution to the second solution is:

11st solution: 2nd solution=2:1

Example 29: Profit Sharing in a Partnership

A, B, and C are three partners in a firm.

- A **i**nvests Rs. 5000 for 6 months**.**
- B invests Rs. 7500 for 8 months**.**
- C invest**s** Rs. 10000 for 10 months**.**

The profit is to be divided in the ratio of their investments multiplied by the time period.

Step 1: Calculate Each Partner's Investment in Proportion to Time

$$\text{A's Investment} = 5000 \times 6 = 30000$$

$$\text{B's Investment} = 7500 \times 8 = 60000$$

$$\text{C's Investment} = 10000 \times 10 = 100000$$

Step 2: Total Investment

$$\text{Total Investment} = 30000 + 60000 + 100000 = 190000$$

Step 3: Calculate C's Share of Profit

Given that the total profit is Rs. 8750, C's share is calculated as:

$$\text{C's Share} = \left(\frac{100000}{190000}\right) \times 8750$$

$$\text{C's Share} = 0.5263 \times 8750 = 4605.26$$

Conclusion:

C's share of the profit would be Rs. 4605.26**.**

Example 30: Scale Drawings A scale model of a building is built with a scale of 1:50. If this model is 45 cm tall what is the actual height of the building?

olution: Step 1: Setting Up the Proportion

Let the actual height of the building be x cm**.**

The given scale ratio is:

$$\frac{1}{50} = \frac{45}{x}$$

Step 2: Applying the Fundamental Property of Proportions

Cross-multiplying the terms, we have:

$$1 \times x = 50 \times 45$$
$$x = 2250$$

Step 3: Converting Units

Since **1 meter = 100 cm**, the actual height in meters is:

$$x = \frac{2250}{100} = 22.5 \text{ m}$$

Problems in Application in the Real World

Example 32: Map Scales The distance between two cities on a map with a scale 1:100,000 is 5.5 cm. How many kilometers is the true distance between these cities?

Solution: Step 1: Setting Up the Proportion

Let the actual distance be x cm**.**

The given scale ratio is:

$$\frac{1}{100,000} = \frac{5.5}{x}$$

Step 2: Applying the Fundamental Property of Proportions

Cross-multiplying the terms, we have:

$$1 \times x = 100,000 \times 5.5$$
$$x = 550,000 \text{ cm}$$

Step 3: Converting Units

Since **100 cm = 1 meter** and **1,000 meters = 1 kilometer**, we can convert the distance as follows:

$$x = \frac{550,000}{100,000} = 5.5 \text{ km}$$

Example 33: Medication Dosage A doctor prescribes a patient weighing 70 kg with 250 mg of a medication. What dosage should another patient (weighing 56 kg) receive?

Solution: The dosage is directly proportional to the weight of the patient.

$$\frac{70}{56} = \frac{250}{x}$$

Using cross multiplication:

$$70x = 56 \times 250$$
$$70x = 14000$$
$$x = \frac{14000}{70} = 200$$

Thus, the patient weighing 56 kg should receive 200 mg of the medication.

Alternatively, using a direct formula:

$$\text{New dosage} = 250 \times \frac{56}{70} = 200 \text{ mg}$$

Example 34: Fuel Economy

A car travels 240 miles using 12 gallons of fuel. Since the car consumes 1 gallon of fuel for every 20 miles, we can determine the fuel required for a 400-mile journey by dividing the total distance by 20.

olution:

he fuel consumption is directly proportional to the distance traveled. Using the proportion:

$$\frac{12}{x} = \frac{240}{400}$$

Applying the cross-multiplication method:

$$240x = 400 \times 12$$

$$240x = 4800$$

$$x = \frac{4800}{240} = 20$$

Thus, 20 gallons of fuel will be required to travel 400 miles..

Example 35: Photographic Enlargement A 4×6 inch photograph is to be enlarged so that its longer side is 15 in. How big will the enlarged photograph be?

Solution:

The enlargement factor is directly proportional to the dimensions.

$$\frac{6}{15} = \frac{4}{x}$$

Using cross-multiplication:

$$6x = 15 \times 4$$

$$6x = 60$$

$$x = \frac{60}{6} = 10$$

Thus, the dimensions of the enlarged photograph will be 10 × 15 inches.

Example 36: Currency Conversion Suppose that 1 USD is equal to 0.85 EUR, how many USD are equal to 510 EUR?

The amount in USD is proportional to the amount in EUR.

$$\frac{1}{x} = \frac{0.85}{510}$$

Using cross-multiplication:

$$1 \times 510 = 0.85 \times x$$

$$510 = 0.85x$$

$$x = \frac{510}{0.85} = 600$$

510 EUR = 600 USD

Example 37: Increase in salary A person gets a salary raise from $3000 to $3300 monthly. What is the percent increase?

Solution: The percentage increase can be calculated by using the formula:

$$\text{Percentage Increase} = \left(\frac{\text{Increase}}{\text{Original Amount}}\right) \times 100\%$$

Increase = $3300 - $3000 = $300

$$\text{Percentage Increase} = \left(\frac{300}{3000}\right) \times 100\% = 10\%$$

Thus, the increase in salary is 10%!

Example 38: Population Growth The population of a city grows from 240 000 to 264 000 over 5 years. If the population continues to grow at the same rate, what will the population be after 8 more years?

Solution:

The increase in population over the 5 year:

264,000 - 240,000 = 24,000.

So the annual growth rate

$$\frac{24,000}{5} = 4,800 \text{ persons per year}$$

opulation after 8 more years:

$$264{,}000+(4{,}800\times 8)$$

$$=264{,}000+38{,}400$$

$$=302{,}400$$

Example 39: Body Mass Index (BMI) A person's BMI can be calculated using the formula BMI = weight (kg) ÷ [height (m)]². What is the height of 72 kg of a person with a BMI of 25?

Using the BMI formula:

$$25 = \frac{72}{h^2}$$

$$h^2 = \frac{72}{25} = 2.88$$

$$h = \sqrt{2.88} = 1.697$$

Therefore, the person's height is approximately 1.7 meters.

Example 40: Calculate Tax — A product has a price of $80 plus tax. What is the total cost including tax if the tax rate is 8 %?

Solution:

Step 1: Calculate the tax amount

$$\text{Tax} = 0.08\times 80 = 6.40$$

Step 2: Add the tax to the product price

$$\text{Total Cost} = 80+6.40 = 86.40$$

Thus, the final amount including tax is $86.40..

Challenging Problems

Example 41: Compound Proportions

If 10 women take 12 days, working 8 hours per day to build the wall, how many days will it take to build a wall that is 1.5 times as large wall for 15 women, working 6 hours per day?

Answer:

Let the required number of days be **x**. The number of days is:

- **Inversely proportional** to the number of women
- **Directly proportional** to the size of the wall
- **Inversely proportional** to the number of hours worked per day

Using proportionality:

$$\frac{12}{x} = \frac{(10 \times 1 \times 6)}{(15 \times 1.5 \times 8)}$$

$$\frac{12}{x} = \frac{60}{180} = \frac{1}{3}$$

$$x = 12 \times \frac{1}{3} = 4$$

Thus, it will take 4 days to complete the larger wall with 15 women working 6 hours per day.

Example 42: Weighted Averages A group consists of 25 students with an average of 72 scored on a test. When 5 more students who have an average score of 80 joins the class, what will be the new average score of the class?

Solution:

The total score of the initial 25 students:

$$25 \times 72 = 1800$$

The total score of the 5 new students:

$$5 \times 80 = 400$$

The new total score:

$$1800 + 400 = 2200$$

The new total number of students:

$$25+5=3025 + 5 = 3025+5=30$$

he new average score:

$$x = \frac{2200}{30} = 73.33$$

Thus, the new average score of the class is 73.33.

UNIT 3PERCENTAGE: A COMPREHENSIVE GUIDE

The percentage is a basic mathematical concept that represents any number in the form of 100. The word “percent” derives from Latin “per centum”—by the hundred. Using percentages allows for a more uniform way of expressing proportional relationships, facilitating comparisons across multiple situations. A percentage is a dimensionless ratio and is a number followed by the percent sign (%). In other words, 25% = 25 out of 100 = 25/100 = 0.25.

Conversions: There are three main types of conversions involving percentages:-

1. **Fraction to Percentage**: Multiply the fraction by 100 and add the percent symbol.

   Example: 3/4 = 3/4 × 100% = 75%

2. **Decimal to Percentage**: Multiply the decimal by 100 and add the percent symbol.

   Example: 0.35 = 0.35 × 100% = 35%

3. **Percentage to Fraction**: Divide the percentage by 100 and express as a fraction in its simplest form.

   Example: 20% = 20/100 = 1/5

4. **Percentage to Decimal**: Divide the percentage by 100.

   Example: 75% = 75/100 = 0.75

Basic Calculations

Finding a Percentage of a Number: To find a percentage of a number, multiply the number by the percentage expressed as a decimal or fraction

- Formula:

$$P\% \text{ of } X = \left(\frac{P}{100}\right) \times X$$

Example: Find 15% of 80

Solution:

15% of 80 = (15/100) × 80 = 0.15 × 80 = 12

Finding What Percentage One Number is of another: To determine what percentage A is of B, divide A by B and multiply by 100.

• **Formula:**

$$\text{What percent of } Y \text{ is } X? = \left(\frac{X}{Y}\right) \times 100\%$$

- **Example:** What percent is 24 of 60?

$$\left(\frac{24}{60}\right) \times 100 = 0.4 \times 100 = 40\%$$

Finding the Original Number When a Percentage is known: If you know the percentage of a number and want to find that number, you divide the amount by the percentage (in decimal or fraction form).

- **Formula:**

$$X = \frac{Y}{(P/100)} \quad (\text{if } Y = P\% \text{ of } X)$$

- **Example:** 35% of a number is 28. What is the number?

Solution:

28 ÷ (35/100) = 28 ÷ 0.35 = 80

Increase and Decrease Percentage: So percentage increase and percentage decrease are very useful in all fields and can be used in finance, economics and statistics to describe how quantities change over time.

Percentage Increase: Proportional increase of a value compared to the initial value, in a percentage.

Formula:

$$\% \text{ Increase} = \left(\frac{\Delta}{O}\right) \times 100\%$$

(where **Δ** is the change in value and **O** is the original value)

Example: the price of a particular shirt rises from $40 to $50. What would be the percentage increase?

Solution:

$$\left(\frac{50-40}{40}\right) \times 100 = \left(\frac{10}{40}\right) \times 100 = 25\%$$

Percentage Decrease: Percentage loss is the loss of a value expressed as a percentage of the original value.

Formula:

$$\% \text{ Decrease} = \left(\frac{\text{Old Value} - \text{New Value}}{\text{Old Value}}\right) \times 100\%$$

Successive Percentage Changes: The following is a common situation: When percentage changes happen sequentially, the aggregate percentage change is different from the simple sum of the two percentage changes. Instead, we have to multiply the factors.

Formula: If there are successive percentage changes of P and Q, the overall percentage change is:

[1 + (P/100)] × [1 + (Q/100)] - 1] × 100%

Example: A stock increases by 20% and then decreases by 10%. What is the overall percentage change?

Solution: [1 + (20/100)] × [1 - (10/100)] - 1] × 100%

[1.2 × 0.9 - 1] × 100% = 1.08 - 1] × 100% = 8%

The stock has increased by 8% overall, not 10% (20% - 10%).

Reverse Percentage: Sometimes we need to find the original value when we know the final value after a percentage change.

- **Formula:**

$$\text{Original Value} = \frac{\text{Final Value}}{[1 + (P/100)]}$$

(P is positive for an increase and negative for a decrease.).

Example: After a 15% increase, a product costs $69. What was the original price?

Solution:

$$\text{Original Price} = \frac{69}{1 + (15/100)} = \frac{69}{1.15} = 60$$

Thus, the original price was $60.

Profit & Loss Applications: Profit and loss calculations are essential in business and personal finance. They often involve percentage calculations to express the relationship between costs, revenues, and profits.

Basic Definitions

- **Cost Price (CP)**: The price at which an article is purchased.
- **Selling Price (SP)**: The price at which an article is sold.
- **Profit**: When SP > CP, the difference is profit.
- **Loss**: When CP > SP, the difference is loss.

Profit and Loss Percentage: Profit and loss percentages are always calculated with respect to the cost price, unless otherwise specified.

Profit Percentage: (Profit / Cost Price) × 100%

Loss Percentage: (Loss / Cost Price) × 100%

Example: A shopkeeper buys a chair for \$80 and sells it for \$100. What is the profit percentage?

Solution:

$$\text{Profit} = \text{Selling Price} - \text{Cost Price}$$
$$= 100 - 80 = 20$$
$$\text{Profit Percentage} = \left(\frac{\text{Profit}}{\text{Cost Price}}\right) \times 100$$
$$= \left(\frac{20}{80}\right) \times 100 = 25\%$$

Thus, the profit percentage is 25%.

Example: A retailer buys a shirt for \$50 and sells it for \$40. What is the loss percentage?

Solution:

$$\text{Loss} = \text{Cost Price} - \text{Selling Price}$$
$$= 50 - 40 = 10$$
$$\text{Loss Percentage} = \left(\frac{\text{Loss}}{\text{Cost Price}}\right) \times 100$$
$$= \left(\frac{10}{50}\right) \times 100 = 20\%$$

Thus, the loss percentage is 20%.

Marked Price and Discount

- **Marked Price (MP)**: The price marked on an article.
- **Discount**: The reduction given on the marked price.
- **Discount Percentage**:

$$\left(\frac{\text{Discount}}{\text{Marked Price}}\right) \times 100\%$$

Example: A store offers a 20% discount on a jacket marked at $100. What is the selling price?

Solution:

$$\text{Discount} = 20\% \text{ of } 100 = \left(\frac{20}{100}\right) \times 100 = 20$$

$$\text{Selling Price} = 100 - 20 = 80$$

Thus, the selling price is $80.

Cost Price when Selling Price and Profit/Loss Percentage are Known

- If profit percentage is P%, then

$$\text{Cost Price (CP)} = \text{Selling Price (SP)} \times \left(\frac{100}{100 + P}\right)$$

- If loss percentage is L%, then

$$\text{Cost Price (CP)} = \text{Selling Price (SP)} \times \left(\frac{100}{100 - L}\right)$$

Example: A merchant sells a watch at $132 and makes a profit of 10%. What is the cost price?

Solution:

$$CP = 132 \times \left(\frac{100}{100 + 10}\right) = 132 \times \left(\frac{100}{110}\right) = 120$$

Example: A shopkeeper sells a bag at $85 and incurs a loss of 15%. What is the cost price?

Solution:

$$CP = 85 \times \left(\frac{100}{100 - 15}\right) = 85 \times \left(\frac{100}{85}\right) = 100$$

Thus, the cost price is $100.

Numerical Problems on Percentage

Problem 1 Express 7/25 as a percentage.

Solution:

$$\frac{7}{25} \times 100 = 28\%$$

Problem 2: 0.0625 as a percentage

Solution:

$$0.0625 \times 100 = 6.25\%$$

Problem 3: Convert 137.5% to decimal and fraction in the simplest form.

Solution:

Decimal Form:

$$137.5\% = \frac{137.5}{100} = 1.375$$

Fraction Form:

$$137.5\% = \frac{1375}{1000} = \frac{11}{8}$$

Problem 4: What is a number if 24% of it is 42?

Solution: Let x be the number.

$$24\% \text{ of } x = 42$$

$$0.24x = 42$$

$$x = \frac{42}{0.24} = 175$$

Thus, the number is 175.

Problem 5: What percentage of 250 is 75?

Solution:

$$\left(\frac{75}{250}\right) \times 100 = 30\%$$

Problem 6: Find 22.5% of 840.

Solution:

$$0.225 \times 840 = 189$$

Problem 7: What is 70% of the same number, if 35% of the number is 105?

Solution:

Step 1: Let the number

$$x.35\% \text{ of } x = 105$$

$$0.35x = 105$$

$$x = 105 \div 0.35 = 300$$

Step 2: 70% of the number.

$$70\% \text{ of } 300 = 0.7 \times 300 = 210$$

Thus, 70% of the number is 210.

Problem 8: Two numbers are in the ratio of 3:5. What is that smaller number as a percentage of the larger number?

Solution: Let the two numbers be 3x and 5x.

$$\left(\frac{3x}{5x}\right) \times 100 = \left(\frac{3}{5}\right) \times 100 = 60\%$$

Problem 9: A mixture consists of milk and water in the ratio 4:1. Answer: 80% of the mixture is water.

Solution:

- Total parts = 4 + 1 = 5
- Water parts = 1

$$\text{Percentage of water} = (1/5) \times 100\% = 20\%$$

Problem 10: In an election, A scored 45% of total votes and B achieved 38%. The rest of votes cast were invalid votes as per results. How many votes were invalid?

Solution:

Invalid votes percentage 100% − 45% − 38%

= 17%

Thus, 17% of the votes were invalid.

Increase and Decrease Percentage

Problem 11: The price of a shirt went up from $40 to $54. All you need is the delta to be divided by the small number that adds up.

Solution:

$$\text{Percentage Increase} = \left(\frac{54-40}{40}\right) \times 100\% = 35\%$$

Problem 12: A town had a population of 25,000 that decreased to 22,500. What is the percentage reduction?

Solution:

$$\left(\frac{25,000-22,500}{25,000}\right) \times 100\% = 10\%$$

Problem 13: After increasing 12%, the price of a laptop is $896. What was the original price?

Solution:

$$x + 0.12x = 1.12x = 896$$

$$x = \frac{896}{1.12} = 800$$

Problem 14: The price of a TV is reduced by 15% to $425. What was the original price?

Solution:

$$0.85x = 425$$

$$x = \frac{425}{0.85} = 500$$

Problem 15: A car loses 20% of its value the first year and 15% the second year. How much overall percentage depreciation happens over two years?

Solution:

$$\text{Final Value} = \text{Original} \times 0.8 \times 0.85 = \text{Original} \times 0.68$$

$$\text{Total Depreciation} = (1 - 0.68) \times 100\% = 32\%$$

Problem 16: A stock rose 25% in price then 20% in price fell. What is the net percentage change?

Solution:

$$\text{Final Value} = \text{Original} \times 1.25 \times 0.8 = \text{Original}$$

$$\text{Overall Change} = 0\%$$

Problem 17: In the first year, a person's salary was increased by 15% and during the second year by 10%. What one percentage point increase would achieve the same effect?

Solution:

Salary = Original × 1.15 × (1 + 0.1) = Original × 1.15 × 1.1 = Original × 1.265

Increase = (1.265 − 1) × 100% = 26.5%

Problem 18: A quantity decreases by 20%. How much percentage should it be increased to return to its original value?

Solution:

$$\frac{20}{80} \times 100\% = 25\%$$

Problem 19: The number of students in a class increased 20% from last year. This year, how many students? 36 students 36 students this year, how many students last year?

Solution:

$$x + 0.2x = 1.2x = 36$$

$$x = 36 \div 1.2 = 30$$

Problem 20: To this extent, a company had its revenue go up by 15% and expenses by 10%. What should be the percentage increase in profit if the revenue was $200,000 and expenses $180,000?

Solution:

$$\text{Profit Increase} = \left(\frac{32,000 - 20,000}{20,000}\right) \times 100\% = 60\%$$

Profit & Loss Applications

Problem 21: A merchant purchases a table at the price of 240 dollars and sells it at the price of 300 dollars. What is the percentage of profit?

Solution:

Profit = Selling Price - Cost Price

= **$300 - $240** = **$60**

Profit Percentage = **(**Profit / Cost Price) × 100%

= **(60 / 240) × 100%** = **25%**

Problem 22: When a merchant sells a watch for $144, he experiences a 20% loss. What was the cost price?

Solution:

Let the cost price be x.

Selling Price (SP) = Cost Price (CP) - 20% of CP
= 0.8x = $144

Cost Price = $144 ÷ 0.8 = $180

Problem 23: Marks Price: It is a price which is 30% more than its cost price. What is his profit percent, if he gives discount of 10% on the marked price?

Solution:

Let the cost price be $100.

Marked Price (MP) = $100 + 30% of $100 = $130

Selling Price (SP) after 10% discount = $130 - 10% of $130
= $130 - $13 = $117

Profit = $117 - $100 = $17

Profit Percentage = (Profit / Cost Price) × 100%
= (17 / 100) × 100% = 17%

Problem 24: However a shopkeeper makes 20% profit selling an item. What will be the profit percentage if the cost price had been 10% lesser and the selling price 10% more?

Solution:

Let the original cost price be $100.

Original Selling Price (SP) = $100 + 20% of $100 = $120

New Cost Price = $100 - 10% of $100 = $90

New Selling Price = $120 + 10% of $120 = $132

New Profit = $132 - $90 = $42

New Profit Percentage = (42 / 90) × 100% = 46.67%

Problem 25: A persons purchase a car for $40,000 and sold it with a loss of 15% He buys another car and sells it for a profit of 25% with the money he receives. What is his overall profit/loss %?

Solution:

First Car:
Cost Price = $40,000
Selling Price = $40,000 - 15% of $40,000
= $40,000 - $6,000 = $34,000
Loss = $6,000

Second Car:

Cost Price = $34,000

Selling Price = $34,000 + 25% of $34,000

= $34,000 + $8,500 = $42,500

Profit = $8,500

Final Profit Calculation:

Final Amount = $42,500

Initial Amount = $40,000

Net Profit = $42,500 - $40,000 = $2,500

Profit Percentage = (2,500 / 40,000) × 100% = 6.25%

Problem 26: Question: A shopkeeper marks an item and gives a 10% discount on the marked price and earns a profit of 20%. B) $200.00

Solution:

Let the marked price be **x**.

Selling Price after 10% discount = 0.9x

Since Profit = 20% of Cost Price:

0.9x - 200 = 0.2 × 200

0.9x - 200 = 40

0.9x = 240

x = 240 ÷ 0.9 = $266.67

Problem 27: There is a shopkeeper who marks his goods 40% above the cost price and gives 20% discount on the market price. How much is his percentage profit?

Solutions:

Let Cost Price = **$100**

Marked Price (MP) = $100 + 40% of $100 = $140

Selling Price after 20% discount = $140 - 20% of $140

= $140 - $28 = $112

Profit = \$112 - \$100 = \$12

Profit Percentage = (12 / 100) × 100% = 12%

Problem 28: One sold two goods at the same price. On one he made 20% profit and on the other he suffered 20% loss. What is his net profit or loss %?

Solutions:

Let the selling price of each item be \$100.

First Item (20% Profit):
Cost Price = \$100 ÷ 1.2 = \$83.33

Second Item (20% Loss):
Cost Price = \$100 ÷ 0.8 = \$125

Total Cost Price = \$83.33 + \$125 = \$208.33
Total Selling Price = \$100 + \$100 = \$200

Total Loss = \$208.33 - \$200 = \$8.33

Loss Percentage = (8.33 / 208.33) × 100% = 4%

Problem 29: A shopkeeper sells an article at 15% profit. If you had purchased it for 10% less, and sold it for \$3 more, you'd have made 30% (profit). What will be cost price of article?

Solution:

Let the cost price be x.

Selling Price = 1.15x

If the cost price was 10% less: 0.9x
If the selling price was \$3 more: 1.15x + 3

New Profit Percentage = 30%

$$(1.15x + 3 - 0.9x)/0.9x = 0.3$$

$$(0.25x + 3)/0.9x = 0.3$$

$$0.25x + 3 = 0.27x$$

$$3 = 0.02x$$

$$x = 3/0.02 = 150$$

Problem 30A trader marks up his goods 35% over the cost price and then allows a discount of 15% on the marked price. A customer purchased an article for $1,147.50. What was the cost price?

Solution:

Let the cost price be x.

Marked Price = 1.35x

Selling Price = 1.35x - 15% of 1.35x
= 1.35x - 0.2025x
= 1.1475x

Given 1.1475x = $1,147.50

X = 1,147.50/1.1475 = 1,000

Cost Price = $1,000

Advanced Percentage Problems

Mixed Applications

Problem 31: 40% of the students in a class are girls. The percentage of girls becomes 50% if 10 more girls join the class. How many students were there to begin with in the class?

Solution:

Let the original number of students be **x**.

- Number of girls initially = 0.4x
- Number of boys = x - 0.4x = 0.6x
- After 10 more girls join:
- New number of girls = 0.4x + 10

- New total number of students = x + 10

Given that the percentage of girls now becomes 50%, we set up the equation:

$$\frac{0.4x + 10}{x + 10} = 0.5$$

Multiplying both sides by (x + 10):

$$0.4x + 10 = 0.5(x + 10)$$
$$0.4x + 10 = 0.5x + 5$$
$$10 - 5 = 0.5x - 0.4x$$
$$5 = 0.1x$$
$$x = 50$$

Thus, the original number of students was **50**.

Problem 32: 72 % of the students passed in Mathematics and 64% of students passed in English in one examination. Implying that since 52% passed both subject, Thus 100 - 52 = 48% failed both subject.

Solution:

Let the total number of students be 500.

- Students who passed Mathematics = 72% of 500 = 360
- Students who passed English = 64% of 500 = 320
- Students who passed both subjects = 52% of 500 = 260

Using the formula:

$$n(M \cup E) = n(M) + n(E) - n(M \cap E)$$
$$n(M \cup E) = 360 + 320 - 260 = 420$$

Students who failed both subjects:

$$500 - 420 = 80$$

Thus, 80 students (or 16%) failed both subjects.

roblem 33: A solution consists of 30% alcohol. 150 ml of this solution ecreases the content of alcohol be 25 % less, how much more water should be added to it?

Solution:

Let **x** be the amount of water to be added.

- Alcohol in the original solution = 30% of 150 = 45 mL
- Total volume after adding water = (150 + x) mL

New concentration equation:

$$\frac{45}{150+x} = 0.25$$
$$45 = 0.25(150 + x)$$
$$45 = 37.5 + 0.25x$$
$$7.5 = 0.25x$$
$$x = 30$$

.

Thus, 30 mL of water should be added.

Problem 34: If his salary increases by 20% and his expenditure increases by 10%, the percentage increase in his savings is:

Solution:

Assume the original salary is $100.

- Initial expenditure = 75% of $100 = $75
- Initial savings = $100 - $75 = $25

After increase:

- New salary = $100 + 20% of $100 = $120
- New expenditure = $75 + 10% of $75 = $82.5
- New savings = $120 - $82.5 = $37.5

Percentage increase in savings:

$$\frac{37.5 - 25}{25} \times 100 = 50\%$$

Thus, savings increase by **50%**.

Problem 35: A certain recipe requires a mixture of two ingredients A and B which are used in the ratio of 3:5. If the quantity of ingredient A is increased by 40%, and if the quantity of ingredient B is decreased by 15%, what is the percentage change in the ratio A:B?

Solution:

Let initial quantities be A = 3x and B = 5x.

After changes:

- New A = 3x + 40% of 3x = 4.2x
- New B = 5x - 15% of 5x = 4.25x

New ratio:

$$\frac{4.2}{4.25} = 0.988 : 1$$

Original ratio:

$$\frac{3}{5} = 0.6 : 1$$

Percentage change:

$$\frac{0.988 - 0.6}{0.6} \times 100 = 64.67\%$$

Thus, the percentage change in the ratio is 64.67%.

Problem 36: During the first year, a company's profits rose by 20 percent and they fell by 10 percent during the second year. Currently, in the third year, the profit grew by 15 percent. What is the overall percentage increase in profit over the three years?

Solution:

Let the initial profit be P.

- After the first year: 1.2P
- After the second year: 1.2P - 10% of 1.2P = 1.08P
- After the third year: 1.08P + 15% of 1.08P = 1.242P

Overall percentage change:

$$\frac{1.242P - P}{P} \times 100 = 24.2\%$$

Thus, the total profit increase over three years is 24.2%.

Problem 37: A thing is a product, you sell it for som… The profit decreases by 20% if the cost price is increased by 15% and the selling price is increased by 10%. 4. What was her original profit percentage?

Let the original cost price be C and selling price be S.

Original profit percentage:

$$\frac{S - C}{C} \times 100$$

After changes:

- New cost price = 1.15C
- New selling price = 1.1S

Given that profit decreased by 20%:

$$1.1S - 1.15C = 0.8(S - C)$$

$$S = \frac{7}{6}C$$

$$\text{Profit} = \frac{S - C}{C} \times 100 = \frac{C/6}{C} \times 100 = 16.67\%$$

Thus, the original profit percentage was 16.67%.

Problem 38: A shopkeeper offers a discount of 10% on the labeled price of an article and then offers an additional discount of 5% on the discounted price. The marked price was at what if the selling price/ the selling price is $171?

Solution:

Let the marked price be **x.**

After the first discount:

$$\text{New price} = x - 10\% = 0.9x$$

After the second discount:

$$\text{Final price} = 0.9x - 5\%$$

Of $0.9x = 0.855x$

Given that **0.855x = 171**, solving for x:

$$x = \frac{171}{0.855} = 200$$

Thus, the original marked price was $200.

Problem 39: A mixture of 60 liters contains a ratio of 2:1 of milk to water. If we take x mL of water in this mixture so that the ratio of milk and water become 1:1

Solution:

- Volume of milk in the original mixture = $\frac{2}{3} \times 60 = 40$ liters
- Volume of water in the original mixture = $\frac{1}{3} \times 60 = 20$ liters
- To make the ratio 1:1, the volume of water must be equal to the volume of milk, i.e., 40 liters.
- Additional water required = $40 - 20 = 20$ liters

Problem 40: The first operation is an increase of 20%, and the second is a decrease of 25%. What percentage has enjoyed an overall change?

Solution:

- Let the initial value be x
- After a 20% increase: $x + 0.2x = 1.2x$
- After a 25% decrease: $1.2x - 0.3x = 0.9x$
- Overall percentage change:

$$\frac{0.9x - x}{x} \times 100 = -10\%$$

- The quantity is reduced by 10%

Real-World Applications

Problem 41: Compound interest of 8% p.a is offered by a bank. how much you will have after 3 years if you deposit $5,000

Solution:

- Formula: $A = P(1+r)^t$
- $A = 5000 \times (1.08)^3$
- $A = 5000 \times 1.2597 = 6298.56$
- Final Amount: $6,298.56

Problem 42: Car depreciation — 15% per year Assuming the car's original price is $25,000, what will it's price be in 4 years?

Solution:

- Formula: $P = P_0 \times (1-r)^t$
- $P = 25000 \times (0.85)^4$
- $P = 25000 \times 0.5220 = 13,050$
- Final Price: $13,050

Problem 43: A city's population grows at 5% annually. If the current population is 200,000, what will it be in 6 years?

Solution:

- Formula: $P = P_0(1+r)^t$
- $P = 200000 \times (1.05)^6$
- $P = 200000 \times 1.3401 = 268,020$
- Final Population: 268,020

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Problem 44: Sales of a company increased 8% in Q1, decreased 3% in Q2, increased 5% in Q3 and increased 2% in Q4. How did sales change, overall, for the year?

Solution:

- Overall factor:

$$(1.08)\times(0.97)\times(1.05)\times(1.02)=1.123$$

- Overall percentage change:

$$(1.123-1)\times100=12.3\%$$

- Final Answer: 12.3% increase

Problem 45: Assume a store has a "Buy 2, Get 1 Free" offer What is the effective percentage discount on the entire purchase?

Solution:

- You buy 3 items but pay for 2
- Discount = $\frac{1}{3}\times 100=33.33\%$
- **Effective Discount: 33.33%**

Problem 46: A salesperson is paid a base salary of $3,000 a month, plus a 5% commission on sales. How much must the salesperson sell in a month if they desire to earn $5,000?

Solution:

- Total earnings formula:

$$5000=3000+0.05\times \text{Sales}$$

- $2000=0.05\times \text{Sales}$
- $\text{Sales}=\frac{2000}{0.05}=40,000$
- Required Sales: $40,000

Problem 47: A product is labelled "20% extra free." If the initial mass was 250 g, what is the new weight?

Solution:

- New weight = Old weight + 20% of old weight
- 250+0.2×250=250+50=300 g
- Final Weight: 300 g

Problem 48: The bill at a restaurant amounts to $120. How much total should you pay if you want to leave a 15% tip?

Solution:

Tip amount = 15% of $120

0.15×120=18

Total payment = 120+18=138

Final Payment: $138

Problem 49: A clothing store is offering a "Buy one, get the second 50% off." If you buy two $40 shirts, what is your effective percentage discount on your just-purchased clothes?

Solution:

- Regular price for two shirts: $40 + $40 = $80
- Sale price: $40 + ($40 \times 0.5) = $40 + $20 = $60
- Discount amount: $80 - $60 = $20
- Percentage discount:

$$\frac{20}{80} \times 100 = 25\%$$

- Final Discount: 25%

UNIT 4 DISCOUNT: COMPREHENSIVE GUIDE

A discount is a reduction in the price of a good or service a seller, or a buyer. This is a standard pricing strategy utilized by many businesses to encourage purchases, clear out inventory, encourage customer loyalty or market to competitors. Mathematically, discount is the deduction of some fixed price from the original price up to which we go to finalize the selling price.The ability to calculate discounts is a critical life skill that enables consumers to make educated buying decisions and businesses to craft effective pricing strategies. This guide discusses various kinds of discount, methods of calculating marked price and selling price, different real-life applications of discount mathematics,.

Types of Discounts

Discounts can be broadly categorized into two main types:

1. Single Discount: A single discount is a one-time reduction applied to the marked price (original price) of a product or service. It is usually expressed as a percentage of the marked price.

Mathematical Formula:

• If MP = Marked Price and d = discount rate (in percentage)

• Discount Amount = MP x d/100

• Selling Price (SP) = MP − Discount Amount = MP × (1−d/100) = MP × (100−d)/100

Example: A television with a marked price of $500 is offered at a 20% discount.

Discount Amount = $500 × 20/100 = $100

• Selling Price = $500 − $100 = $500 × (1 − 20/100) = $500 × 0.8 = $400

2. Successive Discounts: Multiple percentage discounts (sometimes called compound discounts or series discounts) are successive percentage discounts applied in succession. Each discount is calculated from the price after the previous discount, not the marked price.

Mathematical Formula:

- If MP = Marked Price and discounts are $d_1\%$, $d_2\%$, $d_3\%$, etc.

Final Selling Price (SP) = MP × (1 - d_1/100) × (1 - d_2/100) × (1 - d_3/100) × ...

- Alternatively: SP = MP × (100 - d_1)/100 × (100 - d_2)/100 × (100 - d_3)/100 × ...

Equivalent Single Discount: Two or more successive discounts can be replaced by a single equivalent discount. For two successive discounts d_1% and d_2%, the equivalent single **discount is:**

- d_1% + d_2% - (d_1% × d_2%)/100

Example: A shirt with a marked price of $80 is offered with successive discounts of 25% and 10%.

- After first discount: $80 × (1 - 25/100) = $80 × 0.75 = $60
- After second discount: $60 × (1 - 10/100) = $60 × 0.9 = $54
- Overall discount: $80 - $54 = $26, which is 32.5% of the marked price
- Equivalent single discount: 25% + 10% - (25% × 10%)/100 = 35% - 2.5% = 32.5%

Marked Price and Selling Price Calculations: Calculating Selling Price from Marked Price

Single Discount:

- SP = MP × (1 - d/100)

Multiple Successive Discounts:

- SP = MP × (1 - d_1/100) × (1 - d_2/100) × ... × (1 - d_n/100)

Calculating Marked Price from Selling Price

Single Discount:

- MP = SP ÷ (1 - d/100) = SP × 100/(100 - d)

Multiple Successive Discounts:

- MP = SP ÷ [(1 - d_1/100) × (1 - d_2/100) × ... × (1 - d_n/100)]

Calculating Discount Rate

Single Discount:

- Discount Rate (%) = [(MP - SP) ÷ MP] × 100

3. Equivalent Discount Rate for Successive Discounts:

- For two discounts d_1% and d_2%: Equivalent Rate = $d_1 + d_2 - (d_1 \times d_2)/100$
- For three discounts d_1%, d_2%, and d_3%: Equivalent Rate = $d_1 + d_2 + d_3 - (d_1d_2 + d_2d_3 + d_3d_1)/100 + (d_1d_2d_3)/10000$

Application in Real-Life Scenarios

Retail and Shopping

1. Seasonal Sales: Stores often offer discounts during holidays or end-of-season sales. Calculating the final price after discount helps shoppers determine their savings.
2. Clearance Sales: Retailers apply deep discounts to clear old inventory, sometimes offering multiple successive discounts.
3. Promotional Offers: "Buy one, get one 50% off" or similar offers require discount calculations to determine overall savings.
4. Coupons and Promo Codes: These often provide additional discounts on already reduced items, creating successive discount scenarios.

Business and Finance

1. Trade Discounts: Manufacturers offer discounts to wholesalers and retailers based on purchase volume.
2. Cash Discounts: Businesses offer discounts for prompt payment (e.g., "2/10, net 30" means a 2% discount if paid within 10 days, with the full amount due in 30 days).
3. Quantity Discounts: Bulk purchases often qualify for price reductions based on quantity.
4. Loyalty Programs: Businesses reward repeat customers with special discounts.

Real Estate and Large Purchases

. Property Negotiations: Buyers negotiate discounts on listed property prices.
2. Automobile Purchases: Car dealers offer various discounts including manufacturer rebates, dealer discounts, and special promotion discounts.
3. Early Payment Discounts: Many services offer discounts for paying annual fees upfront rather than monthly.

Numerical Problems and Solutions

Single Discount Problems

Problem: A refrigerator is marked at $1,200 and sold at a discount of 15%. Find the selling price and the discount amount.

Solution:

- Marked Price (MP) = $1,200
- Discount Rate = 15%
- Discount Amount = MP × Discount Rate/100 = $1,200 × 15/100 = $180
- Selling Price (SP) = MP - Discount Amount = $1,200 - $180 = $1,020

Problem 2: A furniture store is offering a 25% discount on all items. If a dining table set's selling price after discount is $675, what was its original marked price?

Solution:

- Selling Price (SP) = $675
- Discount Rate = 25%
- Marked Price (MP) = SP ÷ (1 - Discount Rate/100) = $675 ÷ (1 - 25/100) = $675 ÷ 0.75 = $900

Successive Discount Problems

Problem: A clothing store offers successive discounts of 20% and 15% on a jacket. If the original price of the jacket is $200, what is the final selling price?

Solution:

- Marked Price (MP) = $200
- First Discount = 20%
- Price after first discount = $200 × (1 - 20/100) = $200 × 0.8 = $160
- Second Discount = 15%
- Final Selling Price = $160 × (1 - 15/100) = $160 × 0.85 = $136

Problem: An electronic store offers successive discounts of 30% and 10% on a smart watch. What is the equivalent single discount?

Solution:

- First Discount (d_1) = 30%
- Second Discount (d_2) = 10%
- Equivalent Single Discount =

$$d_1 + d_2 - \frac{(d_1 \times d_2)}{100}$$
$$= 30 + 10 - \frac{(30 \times 10)}{100}$$
$$= 40 - 3 = 37$$

Problem: A store offers successive discounts of 25%, 20%, and 10% on a product. If the final price after all discounts is $54, what was the original marked price?

Solution:

- Selling Price (SP) = $54
- Discount Rates: d_1 = 25%, d_2 = 20%, d_3 = 10%
- Marked Price (MP) =

$$MP = \frac{SP}{(1 - d_1/100) \times (1 - d_2/100) \times (1 - d_3/100)}$$
$$MP = \frac{54}{(1 - 25/100) \times (1 - 20/100) \times (1 - 10/100)}$$
$$MP = \frac{54}{0.75 \times 0.8 \times 0.9}$$
$$MP = \frac{54}{0.54} = 100$$

Problem 9: Find the equivalent single discount for two successive discounts of 40% and 30%.

Solution:

First Discount (d_1) = 40%

• Second Discount (d_2) = 30%

• Equivalent Single Discount =

$$d_1 + d_2 - \frac{(d_1 \times d_2)}{100}$$
$$= 40 + 30 - \frac{(40 \times 30)}{100}$$
$$= 70 - 12 = 58$$

Advanced Discount Problems

Problem: A shopkeeper offers 20% discount on the marked price of a product. What is the selling price if the customer gets a discount of Rs 160?

Solution:

- Discount Rate = 20%
- Discount Amount = $160
- Marked Price: (MP) = Discount Amount/ (Discount Rate/100) = $160/(20/100) = $160/0.2 = $800
- Selling Price (SP) = MP – Discount Amount = $800 − $160 = $640

Problem: A shopkeeper offers discounts of 40% and 25% on the marked price of an article. Given the cost price of the article is $300, calculate its marked price.

Solution:

- Cost Price (CP) = $300
- Selling Price (SP) = CP + 20% of CP = $300 + $60 = $360
- Total Discount = 40% + 25% - (40% × 25%)/100

  = 65% - 10%

  = 55%

- Marked Price (MP) = SP ÷ (1 - 55/100)

  = \$360 ÷ 0.45

  = \$800

Challenging Discount Problems

Problem: (Three shops — same item) Store A has sequential discounts of 50% and 20%. Store B provides two discount rates: 40% and 30%. Shop C has a one-time discount of 60%. However, which shop gives the best deal to the customer?

Solution:

- Shop A: Total discount = 50% + 20% - (50% × 20%)/100 = 70% - 10% = 60%
- Shop B: Discount = 40% + 30% - (40% × 30%)/100 = 70% - 12% = 58%
- Shop C: Individual discount = 60%

Shop A has the lowest equivalent percentage discount at 60%.

Multiple Choice Questions (MCQs)

1. The average of 5 numbers is 42. If one number is removed, the new average becomes 40. What is the removed number?
   a) 50
   b) 52
   c) 48
   d) 42

2. If the ratio of two numbers is 4:7 and their sum is 88, what is the larger number?
   a) 32
   b) 56
   c) 40
   d) 28

3. The price of an item increased from ₹800 to ₹960. What is the percentage increase?
   a) 20%
   b) 25%
   c) 30%
   d) 15%

A shopkeeper marks an article at ₹1200 and offers a discount of 15%. What is the selling price?

a) ₹1020

b) ₹1050

c) ₹980

d) ₹1080

5. If A:B = 5:8 and B:C = 3:4, what is A:C?

   a) 5:12

   b) 15:32

   c) 5:10

   d) 15:24

6. A trader allows a discount of 10% on an article and still makes a profit of 8%. If the cost price is ₹500, what is the marked price?

   a) ₹600

   b) ₹580

   c) ₹540

   d) ₹620

7. The average of three numbers is 45. If the sum of two numbers is 80, what is the third number?

   a) 40

   b) 55

   c) 35

   d) 50

8. A man spends 75% of his income and saves ₹3000. What is his total income?

   a) ₹9000

   b) ₹10000

   c) ₹12000

   d) ₹15000

9. If the price of an article is reduced by 20%, by what percentage must it be increased to restore the original price?
   a) 25%
   b) 20%
   c) 30%
   d) 22%

10. A sum of ₹1500 is divided between A and B in the ratio 2:3. How much does B get?
   a) ₹600
   b) ₹750
   c) ₹900
   d) ₹1000

Unsolved Problems

1. The average of six numbers is 55. If one number is removed, the new average becomes 52. Find the removed number.
2. The income of A and B are in the ratio 5:7, and their expenses are in the ratio 3:5. If A saves ₹4000 and B saves ₹6000, find their incomes.
3. A shopkeeper marks an article at ₹1500 and offers a discount of 12%. What is the selling price?
4. If a number is increased by 25% and then decreased by 20%, find the overall percentage change.
5. A sum of ₹2000 is divided among A, B, and C in the ratio 3:5:7. Find the share of each person.

MODULE II MATRICES AND DETERMINANTS

Structure

OBJECTIVES

- To understand the concept of determinants and their properties.
- To calculate the value of determinants up to the third order.
- To explore the definition and types of matrices.
- To perform operations on matrices, including addition, subtraction, and multiplication.
- To determine the adjoint and inverse of a matrix.
- To solve systems of linear equations using matrices.

UNIT 5Determinants

Determinant is a value that can be computed only for square matrices. This is a scalar which can be derived from the entities of the matrix and imparts conversations about the matrix itself. Determinants are used to solve systems of linear equations, compute the inverse of matrices, and study linear transformations. For a square matrix A, the determinant is denoted det(A) or |A|, and occasionally det A.

Calculation of Determinants

First Order Determinant (1×1 matrix)

For a 1×1 matrix [a], the determinant is simply the value a: |a| = a

Second Order Determinant (2×2 matrix)

For a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is given by:

$$|A| = ad - bc$$

This reasoning can be recalled with regards to the product of the elements in the main diagonal minus the product of the elements in the other diagonal.

Third Order Determinant (3by3 matrics)

An example of a 3×3 matrix:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

There are several ways to calculate the determinant:

Step 1: Cofactor Expansion (Laplace Expansion)

$$a(ei - fh) - b(di - fg) + c(dh - eg)$$

We can express this as: $|A| = a|A11| - b|A12| + c|A13|$

Where |A11|, |A12| and |A13| are the determinants of the 2×2 submatrices created by eliminating the first row and respective columns

Method 2: Rule of Sarrus

Be careful to use aif - it is the Gram determinant |A|: we find that|A| = aei + bfg + cdh - ceg - bdi - afh

This can be seen by rewriting the first two columns next to the original matrix and proceeding on the diagonals:

6. Positive terms: product of elements in the three right-to-left diagonals
7. Pluses: the product of the elements of the three diagonals left to right

Properties of Determinants

Transpose Property: The determinant of a matrix equals the determinant of its transpose. |A| = |A^T|

2. **Multiplicative Property**: The determinant of a product of matrices equals the product of their determinants. |AB| = |A| × |B|
3. **Scalar Multiplication**: When a matrix is multiplied by a scalar k, its determinant is multiplied by k^n, where n is the order of the matrix. |kA| = k^n|A|
4. **Singular Matrix**: A matrix is singular (non-invertible) if and only if its determinant is zero. |A| = 0 ⟺A is singular

1. **Row/Column Operations**:
   - If two rows (or columns) are interchanged, the determinant changes sign.
   - If a row (or column) is multiplied by a scalar k, the determinant is multiplied by k.
   - If a multiple of one row (or column) is added to another row (or column), the determinant remains unchanged.
2. **Triangular Matrix**: For a triangular matrix (upper or lower), the determinant equals the product of the elements on the main diagonal.
3. **Determinant of Identity Matrix**: The determinant of the identity matrix is 1. |I| = 1

Examples

Example: Second Order Determinant

Calculate the determinant of the 2×2 matrix.

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$$

Solution:

$$|A| = (3 \times 1) - (4 \times 2)$$
$$= 3 - 8 = -5$$

Example: Determinant of Order Three

Find the determinant of the following matrix A = [2 0 1] [3 4 2] [1 2 5]

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

Solution (Cofactor Expansion along the First Row):

$$|A| = 2\begin{vmatrix} 4 & 2 \\ 2 & 5 \end{vmatrix} - 0\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$$

Calculating the 2×2 determinants:

$$\begin{vmatrix} 4 & 2 \\ 2 & 5 \end{vmatrix} = (4 \times 5 - 2 \times 2) = 20 - 4 = 16$$

$$\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} = (3 \times 5 - 2 \times 1) = 15 - 2 = 13$$

$$\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = (3 \times 2 - 4 \times 1) = 6 - 4 = 2$$

Now, substituting these values:

$$|A| = 2(16) - 0(13) + 1(2)$$

$$= 32 + 2 = 34$$

Problem: Determinant of matrices is far more important than you think!

Solution by cofactor expansion on first row: |1 2 3| = 1|5 6| — 2|4 6| + 3|4 5|
|4 5 6| |8 9| |7 9| |7 8|

= 1(5 × 9 − 6 × 8) − 2(4 × 9 − 6 × 7) + 3(4 × 8 − 5 × 7) = 1(45 − 48) −

2(36 − 42) + 3(32 − 35) = 1(−3) − 2(−6) + 3(-3) = −3 + 12 − 9 = 0

Problem: Solution via cofactor expansion along the second row (it has

a zero): |4 -2 3|=−1^(2+1)|4 3| + 1^(2+2)|4 -2| + 1^(2+3)|4 -2 | | 0 1−1|

|2 6| |2 5| |0 1|

= -1|4 3| + 1|4 -2| - 1|4 -2| |2 6| |2 5| |0 1|

The third term: |4 -2|= 4 × 1 - (-2) × 0 = 4 |0 1|

First term: |4 3| = 4 × 6 - 3 × 2 = 24 - 6 = 18 |2 6|

Indicating for the second term: | 4 − 2 | = 4 × 5 − (− 2) × 2 = 20 + 4 = 24 | 2 5 |

Final calculation = -1(18) + 1(24) - 1(4) = -18 + 24 - 4 = 2

Problem: When |A| = 5, |B| = -3, find |AB|, |A^T| and |3A|.

Solution:

- |AB| = |A| × |B| = 5 × (−3) = −15
- |A^T| = |A| = 5
- If A is a 2×2 matrix: |3A| = 32×|A| = 9×5 = 45
- If A is a 3×3 matrix: |3A|= 3^3 ×|A| = 27 × 5 = 135

Problem: Show The Determinant Of The Matrix: A = a b c d e f g h i

if a + d + g = 0, b + e + h = 0, and c + f + i = 0.

solution: a + d + g = 0 ⇒ d = -a - g b + e + h = 0 ⇒ e = -b - h c + f + i = 0 ⇒ f = -c - i

Use A = [a b c] [- a - g - b - h - c - i] [g h i]

The second row is now the opposite of the sum of the first and third rows. The determinant is zero when one row is a linear combination of other rows. Therefore, |A| = 0.

Problem: Evaluate:

We are given the determinant of the 3×3 matrix:

$$A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}$$

Step 1: Expanding Along the First Row

Using cofactor expansion along the first row:

$$|A| = 1\begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

Step 2: Evaluating the 2×2 Determinants

1. **First determinant:**

$$\begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} = (2 \times 6) - (3 \times 3) = 12 - 9 = 3$$

2. **Second determinant:**

$$\begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = (1 \times 6) - (3 \times 1) = 6 - 3 = 3$$

3. **Third determinant:**

$$\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = (1 \times 3) - (2 \times 1) = 3 - 2 = 1$$

Step 3: Substituting the Determinants

$$|A| = (1 \times 3) - (1 \times 3) + (1 \times 1)$$

$$= 3 - 3 + 1$$

$$= 1$$

Final Answer

Problem

Let A be a **3×3 matrix**, and we are given:

$$|3A| = 24$$

We use the determinant property:

$$|kA| = k^n \cdot |A|$$

where k is a scalar, and n is the number of rows (or columns) of the matrix. Since A is a **3×3 matrix**, we have:

$$|3A| = 3^3 \cdot |A|$$

$$|3A| = 27|A|$$

Substituting the given value $|3A| = 24$:

$$27|A| = 24$$

olving for $|A|$:

$$|A| = \frac{24}{27} = \frac{8}{9}$$

Thus, the determinant of A is $\frac{8}{9}$.

Final Answer:

$$|A| = \frac{8}{9}$$

Problem: Evaluate

We are given the determinant of the 2×2 matrix:

$$A = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

Step 1: Compute the Determinant

Using the determinant formula for a 2×2 matrix:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = (a \times d) - (b \times c)$$

Substituting the given values:

$$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = (\cos\theta \times \cos\theta) - (-\sin\theta \times \sin\theta)$$

$$= \cos^2\theta + \sin^2\theta$$

Using the fundamental trigonometric identity:

$$\cos^2\theta + \sin^2\theta = 1$$

UNIT 6Matrices

A matrix is a rectangular array of numerical or symbols or expressions arranged in rows and columns. For now, understand: Matrices are mathematical objects: they are used everywhere from linear algebra to computer graphics, from physics to economics to this is what you'll hear most science and engineering. The matrix with m rows and n columns is called an m × n matrix, and the integers m and n together are referred to as the dimensions of the matrix. Entries or elements of the matrix are the numbers, symbols, or expressions contained in the matrix.

For instance, a 2 × 3 matrix A can be expressed as:

A = [a11 a12 a13] [a21 a22 a23]

with aij as the element in the ith row and jth column.

Types of Matrices

As far as their dimensions, elements and specific properties matrices can be of various types. Here are some common types:

Square Matrix

Square Matrix: In a case where a matrix has equal number of rows and columns (i.e., m = n). For example:

[2 3] [4 5]

is a 2 × 2 square matrix.

Rectangular Matrix

Rectangular/Non-square Matrix: A matrix containing a different number of rows and columns(m ≠ n). For example:

[1 2 3] [4 5 6]

is a rectangle (2 × 3) matrix.

Row Matrix

Row Matrix: A matrix which contains a single row is called a row matrix. For example:

[1 2 3 4]

is a 1 × 4 row matrix.

Column Matrix

A column matrix is a matrix which has only a single column. For example:

[1] [2] [3]

is a 3 × 1 column matrix.

Diagonal Matrix

Diagonal Matrix a matrix having all elements zero except elements of main diagonal is called as Diagonal Matrix. For example:

[3 0 0] [0 7 0] [0 0 9]

Identity Matrix

An identity matrix is a diagonal matrix in which every diagonal element is equal to 1, which is denoted by I. For example:

[1 0 0] [0 1 0] [0 0 1]

is a 3 × 3 identity matrix.

Zero Matrixes

A zero matrix is a matrix with each of its elements as zero. For example:

[0 0] [0 0]

is a 2 × 2 zero matrix.

Upper Triangular Matrix

An upper triangular matrix is a square matrix in which all the entries below the main diagonal are zero. For example:

[3 4 2] [0 7 1] [0 0 9]

Lower Triangular Matrix

A matrix is lowering triangular if and only if it is square and every element above the main diagonal is zero. For example:

[3 0 0] [4 7 0] [2 1 9]

Symmetric Matrix

A matrix that is its own transpose is known as a symmetric matrix. For example:

[2 3 4] [3 5 6] [4 6 8]

Skew-Symmetric Matrix

A matrix A is called skew-symmetric if A^T = -A. For example:

[0 3 -2] [-3 0 1] [2 -1 0]

Orthogonal Matrix

A square matrix A is orthogonal if and only if its transpose A^T equals to its inverse A^(-1). For example:

[0 1] [-1 0]

Singular Matrix: A singular matrix is a square matrix with determinant zero.

Non-Singular Matrix: a non-singular matrix is a square matrix with a non-zero determinant.

Operations on Matrices

Addition and Subtraction: Matrix addition and subtraction are done element wise. Addition and subtraction of matrices: A pair of matrices can only be added or subtracted if they have the same dimensions.

Addition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of the same order, their sum is given by:

$$C = A + B = [c_{ij}]$$

where:

$$c_{ij} = a_{ij} + b_{ij}$$

Example:

Given:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Then:

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Subtraction

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of the same order, their difference is given by:

$$C = A - B = [c_{ij}]$$

where:

$$c_{ij} = a_{ij} - b_{ij}$$

Example:

Given:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Then:

$$A - B = \begin{bmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Scalar Multiplication

Scalar multiplication involves multiplying each element of a matrix by a scalar k.

If:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad k = 3$$

Then:

$$kA = 3 \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 \times 1 & 3 \times 2 \\ 3 \times 3 & 3 \times 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

Matrix Multiplication

Matrix multiplication is not as straightforward as addition or subtraction. If matrix **A** has dimensions **m × n** and matrix **B** has dimensions **n × p**, then their product **C = A × B** is an **m × p** matrix.

The element c_{ij} of matrix **C** is obtained by taking the dot product of the ith row of **A** and the jth column of **B**:

$$c_{ij} = a_{i1} \times b_{1j} + a_{i2} \times b_{2j} + \ldots + a_{in} \times b_{nj}$$

Example:

Given:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Then:

$$A \times B = \begin{bmatrix} 1 \times 5 + 2 \times 7 & 1 \times 6 + 2 \times 8 \\ 3 \times 5 + 4 \times 7 & 3 \times 6 + 4 \times 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Important Note: Matrix multiplication is **not commutative**, meaning that in general, $A \times B \neq B \times A$.

Properties of Matrix Operations

Associative Property

- **Addition:** $(A + B) + C = A + (B + C)$
- **Multiplication:** $(A \times B) \times C = A \times (B \times C)$

Distributive Property

- $A \times (B + C) = A \times B + A \times C$
- $(A + B) \times C = A \times C + B \times C$

Identity Matrices

- **Additive Identity:** $O + A = A$ where O is the zero matrix.
- **Multiplicative Identity:** $A \times I = I \times A = A$, where I is the identity matrix of appropriate size.

Transpose Properties

- $(A + B)^T = A^T + B^T$
- $(A \times B)^T = B^T \times A^T$
- $(kA)^T = kA^T$, where k is a scalar.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by:

$$T(x, y) = (2x + y, x - y)$$

Step 1: Finding the Matrix Representation of T

A linear transformation can be represented in matrix form as:

$$T(x, y) = A \begin{bmatrix} x \\ y \end{bmatrix}$$

where A is the **transformation matrix.**

To determine A, we compute $T(x, y)$ for the standard basis vectors:

1. **Applying T to $(1, 0)$:**

$$T(1, 0) = (2(1) + 0, 1 - 0) = (2, 1)$$

2. **Applying T to $(0, 1)$:**

$$T(0, 1) = (2(0) + 1, 0 - 1) = (1, -1)$$

Thus, the matrix representation of T is:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

Step 2: Computing $T(3, 4)$

Now, we apply T to the input vector $(3, 4)$:

$$T(3, 4) = A \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Performing the matrix multiplication:

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} (2 \times 3) + (1 \times 4) \\ (1 \times 3) + (-1 \times 4) \end{bmatrix}$$

$$= \begin{bmatrix} 6 + 4 \\ 3 - 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \end{bmatrix}$$

Final Answer:

$$T(3, 4) = (10, -1)$$

Example 12: Solving a System of Linear Equations

Solve the following system of linear equations:

$$2x + y = 5$$

$$3x + 2y = 8$$

Solution:

We express the system as a matrix equation:

$$AX = B$$

where:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

From **Example 10**, we already found that:

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

Now, solving for X:

$$X = A^{-1}B$$

$$X = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

Perform matrix multiplication:

$$X = \begin{bmatrix} (2 \times 5) + (-1 \times 8) \\ (-3 \times 5) + (2 \times 8) \end{bmatrix} = \begin{bmatrix} 10 - 8 \\ -15 + 16 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Thus, the solution is:

$$x = 2, \quad y = 1$$

Advanced Topics in Matrix Theory

Eigenvalues and Eigenvectors

v ≠ 0: A square matrix A has an eigenvector v, and the eigenvalue corresponding to v is λ such that Av = λv.

The eigenvalues of A are derived from the eigenvalue equation, where the characteristic equation of A is $\det(A - \lambda I) = 0$.

Diagonalization

A matrix A is diagonalizable if there is an invertible matrix and diagonal matrix D such that P^(-1)AP = D. The columns of P give the eigenvectors and the diagonal entries give the corresponding Eigenvalues.There are still many operations we can perform on our data, such as Singular Value Decomposition (SVD).Any matrix A is decomposed as A = UΣV^T, where U and V are orthogonal, and Σ is diagonal with non-negative real terms. The diagonal entries are known as the singular values of A.

Jordan Normal Form

It is not the case that all matrices are diagonalizable. When you have a non-diagonalizable matrix, the Jordan form is the best you can do to "diagonalize" the matrix in certain senses.

Matrix Functions

Matrix functions, e.g. e^A, sin(A), log(A), can be defined for square matrices. You can then cause separation of variables, that is important to solve systems of differential equations and for different applications.

Applications of Matrices

Linear Transformations: Matrices are the representation of linear transforms in vector spaces. Such as, matrix can describe representative rotations, reflections and projections in 2D and 3D space.

Systems of Linear Equations; Matrices are a compact way to describe and solve systems of linear equations.

Computer Graphics: Matrices are a cornerstone of computer graphics: translation, rotation, scaling, perspective projection all rely on matrix operations.

Physics and Engineering: Matrices help in modeling physical systems, solving differential equations and analyzing physical systems in physics and engineering.

Economics and Finance: In economics and finance, matrices are used in input-output analysis, Markov chains, and portfolio optimization.

Diagrams and Charts for Machine Learning and Data Analysis

NP: represent in a matrix — matrices are essential in machine learning algorithms, dimensionality reduction techniques (e.g. PCA) and in data analysis.

$$A \times B = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

Each element is computed as follows:

Row 1, Column 1: $(2 \times 1) + (-1 \times 3) + (3 \times 5) = 2 - 3 + 1$

Row 1, Column 2: $(2 \times 0) + (-1 \times 1) + (3 \times 6) = 0 - 1 + 1$

Row 1, Column 3: $(2 \times 2) + (-1 \times 4) + (3 \times 7) = 4 - 4 + 2$

Row 2, Column 1: $(0 \times 1) + (4 \times 3) + (5 \times 5) = 0 + 12 + 25$

Row 2, Column 2: $(0 \times 0) + (4 \times 1) + (5 \times 6) = 0 + 4 + 30$

Row 2, Column 3: $(0 \times 2) + (4 \times 4) + (5 \times 7) = 0 + 16 + 35$

Row 3, Column 1: $(6 \times 1) + (7 \times 3) + (8 \times 5) = 6 + 21 + 40$

Row 3, Column 2: $(6 \times 0) + (7 \times 1) + (8 \times 6) = 0 + 7 + 48$

Row 3, Column 3: $(6 \times 2) + (7 \times 4) + (8 \times 7) = 12 + 28 + 5$

Thus,

Properties of Adjoin

1. For any square matrix A, A × adj(A) = adj(A) × A = |A| × I, |A| is the determinant of A, and I is the identity matrix.
2. For 2×2 matrix A = [a b; c d], it holds that adj(A) = [d -b; -c a].
3. For a matrix A which is a non-singular matrix (|A| ≠ 0), the inverse of A can be computed as, A^(-1) = adj(A)/|A|.
4. If the matrix A is singular (|A| = 0), then A × adj(A) = 0, which is a null matrix.
5. For any scalar k and the n × n (square) matrix A, we have: adj(kA) = k^(n-1) × adj(A)
6. For two square matrices A and B of same order, adj(AB) = adj(B) × adj(A)

Inverse of a Matrix: A square matrix A is invertible if there exists a matrix B such that AB = BA = I, where I is the identity matrix. The matrix B is referred to as the inverse of A and is written as A^(-1).

Conditions for Inverse to Exist

A square matrix A is invertible if and only if |A|≠ 0.

Adjoint Method to Calculate the Inverse

If A is a non-singular matrix, we can get the inverse using the formula:

A^(-1) = adj(A)/|A|

Steps to Find the Inverse Using Adjoint

1. Note that this is a 3×3 matrix A.
2. Make sure that you check the determinant and it is not zero. If is zero, the matrix is not invertible.
3. We are given a square matrix A and we are to find the adjoint of A.
4. Use the equation A^(-1) = adj(A)/|A|, to find the inverse.

Properties of Inverse

1. (A^(-1))^(-1) = A
2. (kA)^(-1) = (1/k)A^(-1), where k is a non-zero scalar
3. (AB)^(-1) = B^(-1)A^(-1)
4. (A^T)^(-1) = (A^(-1))^T

Step 4: Compute the Determinant

$$|A| = 3(7) + (-1)(3) + 1(2) = 21 - 3 + 2 = 20$$

Step 5: Compute the Inverse using $A^{-1} = \frac{\text{adj}(A)}{|A|}$

$$A^{-1} = \frac{1}{20}\begin{bmatrix} 7 & 3 & 2 \\ -5 & 2 & 11 \\ 3 & -7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{20} & \frac{3}{20} & \frac{2}{20} \\ \frac{-5}{20} & \frac{2}{20} & \frac{11}{20} \\ \frac{3}{20} & \frac{-7}{20} & \frac{5}{20} \end{bmatrix}$$

$$= \begin{bmatrix} 0.35 & 0.15 & 0.1 \\ -0.25 & 0.1 & 0.55 \\ 0.15 & -0.35 & 0.25 \end{bmatrix}$$

Thus, the **adjoint** of A is:

$$\text{adj}(A) = \begin{bmatrix} 7 & 3 & 2 \\ -5 & 2 & 11 \\ 3 & -7 & 5 \end{bmatrix}$$

And the **inverse** of A is:

$$A^{-1} = \begin{bmatrix} 0.35 & 0.15 & 0.1 \\ -0.25 & 0.1 & 0.55 \\ 0.15 & -0.35 & 0.25 \end{bmatrix}$$

- C22 = (1×(-1)) - (1×4) = -1 - 4 = -5
- C23 = -[(1×1) - (2×4)] = -[1 - 8] = 7
- C31 = [(2×1) - (1×(-2))] = 2 + 2 = 4
- C32 = -[(1×1) - (1×3)] = -[1 - 3] = 2
- C33 = (1×(-2)) - (2×3) = -2 - 6 = -8

Step 2: Construct the Cofactor Matrix

$$C = \begin{bmatrix} -4 & 3 & -1 \\ 2 & 2 & -2 \\ 2 & -4 & 2 \end{bmatrix}$$

Step 3: Compute the Adjoint (Transpose of Cofactor Matrix)

$$\text{adj}(A) = C^T = \begin{bmatrix} -4 & 2 & 2 \\ 3 & 2 & -4 \\ -1 & -2 & 2 \end{bmatrix}$$

Step 4: Compute the Determinant

$$|A| = 2(-4) + 4(2) + 6(2) = -8 + 8 + 12 = 12$$

Step 5: Compute the Inverse using $A^{-1} = \frac{\text{adj}(A)}{|A|}$

$$A^{-1} = \frac{1}{12} \begin{bmatrix} -4 & 2 & 2 \\ 3 & 2 & -4 \\ -1 & -2 & 2 \end{bmatrix}$$

UNIT 7Solution of Linear Equations

Solving Systems of Linear Equations with Unique Solutions Involving up to Three VariablesA linear system is two or more linear equations sharing the same variables. Then each linear equation can be compressed to ax + by + cz +... = d, where a, b, c, d are constants, x, y, z are variables. It means when we solve a system of linear equations, we are actually looking for the values of the variables that make all the equations true at the same time.When there is only one possible value for each of the variables, a system of linear equations is said to have a unique solution. That is when the equations have independence where the number of equations equals the number of variables.

Methods for Solving Systems of Linear Equations

1. Substitution Method: The substitution method involves:-

1. One equation solved for one variable as a function of all the other variables
2. By replacing this expression in the other equations
3. Solve the resulting system with one less variable
4. Resolving substituted terms to find other variabl

2. Elimination Method: The elimination method involves:-

1. Multiply both sides of these equations by constants until the coefficients match
2. Using addition or subtraction of equations to eliminate variables
3. Less simple: Solving the resulting equations having fewer variables
4. Then, we back-substitue to get the value of all the variables

3. Matrix Methods: Matrix methods involve:-

1. Matricial expression of the system
2. Row Echelon form via Elementary row Operations
3. This implies the values of all variables; if we work backwards,

Two-Variable Systems: A system of two linear equations with two variables can be written as:-

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

Where a_1, a_2, b_1, b_2, c_1, and c_2 are constants.

Graphical Interpretation: Graphically, each equation represents a line in the xy-plane. The solution to the system is the point of intersection of these lines. A unique solution exists when the lines intersect at exactly one point.

Three-Variable Systems: A system of three linear equations with three variables can be written as:

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

Where a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, and d_3 are constants.

Graphical Interpretation: Graphically, each equation represents a plane in three-dimensional space. The solution to the system is the point of intersection of these planes. A unique solution exists when the planes intersect at exactly one point.

Examples

Example 1: Two-Variable System using Substitution Method

Solve the system:

$2x + 3y = 8$

$4x - y = 5$

Solution:

1. Solve the second equation for y:
2. $4x - y = 5$
3. $-y = 5 - 4x$
4. $y = 4x - 5$

5. Substitute this expression for y into the first equation:
6. 2x + 3(4x - 5) = 8
7. 2x + 12x - 15 = 8
8. 14x - 15 = 8
9. 14x = 23
10. x = 23/14
11. Back-substitute to find y:
12. y = 4(23/14) - 5
13. y = 92/14 - 5
14. y = 92/14 - 70/14
15. y = 22/14
16. y = 11/7

Therefore, the solution is x = 23/14 and y = 11/7.

Example 2: Two-Variable System using Elimination Method

Solve the system:

3x + 2y = 7

5x - 3y = 1

Solution:

1. Multiply the first equation by 3:
2. 9x + 6y = 21
3. Multiply the second equation by 2:
4. 10x - 6y = 2
5. Add the two equations to eliminate y:
6. 9x + 6y = 21

7. 10x - 6y = 2

8. ---------------

9. 19x = 23

10. Solve for x:

11. x = 23/19

12. Substitute this value back into the first equation:

13. 3(23/19) + 2y = 7

14. 69/19 + 2y = 7

15. 2y = 7 - 69/19

16. 2y = 133/19 - 69/19

17. 2y = 64/19

18. y = 32/19

Therefore, the solution is x = 23/19 and y = 32/19.

Example 3: Three-Variable System using Elimination Method

Solve the system:

x + y + z = 6

2x - y + z = 3

x + 2y - z = 2

Solution:

1. Eliminate z from the first two equations by subtracting the first from the second:

2. x + y + z = 6

3. 2x - y + z = 3

4. ---------------

5. $x - 2y = -3$

6. Eliminate z from the first and third equations by adding them:

7. $x + y + z = 6$

8. $x + 2y - z = 2$

9. ---------------

10. $2x + 3y = 8$

11. Now we have a two-variable system:

12. $x - 2y = -3$

13. $2x + 3y = 8$

14. Solve this system by elimination. Multiply the first equation by 2:

15. $2x - 4y = -6$

16. $2x + 3y = 8$

17. ---------------

18. $-7y = -14$

19. Solve for y:

20. $y = 2$

21. Substitute this value back into $x - 2y = -3$:

22. $x - 2(2) = -3$

23. $x - 4 = -3$

24. $x = 1$

25. Substitute $x = 1$ and $y = 2$ into the first original equation:

26. $1 + 2 + z = 6$

27. $3 + z = 6$

28. $z = 3$

Therefore, the solution is x = 1, y = 2, and z = 3.

Example 4: Three-Variable System using Substitution Method

Solve the system:

2x - y + 3z = 9

x + y + z = 6

3x - 2y - z = 1

Solution:

1. From the second equation, express x in terms of y and z:
2. x + y + z = 6
3. x = 6 - y - z
4. Substitute this expression into the first equation:
5. 2(6 - y - z) - y + 3z = 9
6. 12 - 2y - 2z - y + 3z = 9
7. 12 - 3y + z = 9
8. -3y + z = -3
9. Substitute the expression for x into the third equation:
10. 3(6 - y - z) - 2y - z = 1
11. 18 - 3y - 3z - 2y - z = 1
12. 18 - 5y - 4z = 1
13. -5y - 4z = -17
14. Now we have a two-variable system:
15. -3y + z = -3
16. -5y - 4z = -17
17. Solve for z from the first equation:

18. z = -3 + 3y

19. Substitute this expression into the second equation:

20. -5y - 4(-3 + 3y) = -17

21. -5y + 12 - 12y = -17

22. -17y + 12 = -17

23. -17y = -29

24. y = 29/17

25. Back-substitute to find z:

26. z = -3 + 3(29/17)

27. z = -3 + 87/17

28. z = -51/17 + 87/17

29. z = 36/17

30. Back-substitute to find x:

31. x = 6 - y - z

32. x = 6 - 29/17 - 36/17

33. x = 6 - 65/17

34. x = 102/17 - 65/17

35. x = 37/17

Therefore, the solution is x = 37/17, y = 29/17, and z = 36/17.

Numerical Problems

Problem 1: Solve the system of equations using the substitution method.

3x + 2y = 12

x - y = 3

Solution:

1. From the second equation:
2. x - y = 3
3. x = 3 + y
4. Substitute into the first equation:
5. 3(3 + y) + 2y = 12
6. 9 + 3y + 2y = 12
7. 9 + 5y = 12
8. 5y = 3
9. y = 3/5
10. Back-substitute to find x:
11. x = 3 + 3/5
12. x = 15/5 + 3/5
13. x = 18/5

Therefore, the solution is x = 18/5 and y = 3/5.

Problem 2: Solve the system of equations using the elimination method.

5x + 2y = 16

3x - 4y = 4

Solution:

1. Multiply the first equation by 2:
2. 10x + 4y = 32
3. Multiply the second equation by 1:
4. 3x - 4y = 4

5. Add the equations:
6. 10x + 4y = 32
7. 3x - 4y = 4
8. ---------------
9. 13x = 36
10. Solve for x:
11. x = 36/13
12. Substitute back into the first equation:
13. 5(36/13) + 2y = 16
14. 180/13 + 2y = 16
15. 2y = 16 - 180/13
16. 2y = 208/13 - 180/13
17. 2y = 28/13
18. y = 14/13

Therefore, the solution is x = 36/13 and y = 14/13.

Problem 3: Solve the system of equations using the elimination method.

2x + 3y = 8

4x - 3y = 10

Solution:

1. Add the two equations:
2. 2x + 3y = 8
3. 4x - 3y = 10
4. ---------------

5. 6x = 18
6. Solve for x:
7. x = 3
8. Substitute back into the first equation:
9. 2(3) + 3y = 8
10. 6 + 3y = 8
11. 3y = 2
12. y = 2/3

Therefore, the solution is x = 3 and y = 2/3.

Problem 4: Solve the system of equations using the substitution method.

x - 2y = 5

3x + y = 4

Solution:

1. From the first equation:
2. x - 2y = 5
3. x = 5 + 2y
4. Substitute into the second equation:
5. 3(5 + 2y) + y = 4
6. 15 + 6y + y = 4
7. 15 + 7y = 4
8. 7y = -11
9. y = -11/7
10. Back-substitute to find x:

11. x = 5 + 2(-11/7)

12. x = 5 - 22/7

13. x = 35/7 - 22/7

14. x = 13/7

Therefore, the solution is x = 13/7 and y = -11/7.

Problem 5: Solve the system of equations using the elimination method.

x + y + z = 6

2x - y + z = 2

x + 2y - z = 3

Solution:

1. Subtract the first equation from the second:
2. x + y + z = 6
3. 2x - y + z = 2
4. ---------------
5. x - 2y = -4
6. Subtract the first equation from the third:
7. x + y + z = 6
8. x + 2y - z = 3
9. ---------------
10. y - 2z = -3
11. Now we have:
12. x - 2y = -4
13. y - 2z = -3

14. From the second equation:

15. y - 2z = -3

16. y = -3 + 2z

17. Substitute into the first equation:

18. x - 2(-3 + 2z) = -4

19. x + 6 - 4z = -4

20. x - 4z = -10

21. From the original first equation:

22. x + y + z = 6

23. Substitute the expression for y:

24. x + (-3 + 2z) + z = 6

25. x - 3 + 2z + z = 6

26. x + 3z = 9

27. Now we have:

28. x - 4z = -10

29. x + 3z = 9

30. Subtract the first from the second:

31. 7z = 19

32. z = 19/7

33. Back-substitute to find x:

34. x + 3(19/7) = 9

35. x + 57/7 = 9

36. x = 9 - 57/7

37. x = 63/7 - 57/7

38. x = 6/7
39. Back-substitute to find y:
40. y = -3 + 2(19/7)
41. y = -3 + 38/7
42. y = -21/7 + 38/7
43. y = 17/7

Therefore, the solution is x = 6/7, y = 17/7, and z = 19/7.

Problem 6: Solve the system of equations using the substitution method.

2x + 3y - z = 5

x - y + 2z = 4

3x + y + z = 10

Solution:

1. From the second equation:
2. x - y + 2z = 4
3. x = 4 + y - 2z
4. Substitute into the first equation:
5. 2(4 + y - 2z) + 3y - z = 5
6. 8 + 2y - 4z + 3y - z = 5
7. 8 + 5y - 5z = 5
8. 5y - 5z = -3
9. y - z = -3/5
10. Substitute into the third equation:
11. 3(4 + y - 2z) + y + z = 10

12. 12 + 3y - 6z + y + z = 10

13. 12 + 4y - 5z = 10

14. 4y - 5z = -2

15. Now we have:

16. y - z = -3/5

17. 4y - 5z = -2

18. Multiply the first equation by 4:

19. 4y - 4z = -12/5

20. Subtract from the second equation:

21. 4y - 5z = -2

22. 4y - 4z = -12/5

23. ---------------

24. -z = -2 + 12/5

25. -z = -10/5 + 12/5

26. -z = 2/5

27. z = -2/5

28. Substitute back to find y:

29. y - (-2/5) = -3/5

30. y + 2/5 = -3/5

31. y = -3/5 - 2/5

32. y = -5/5

33. y = -1

34. Substitute back to find x:

35. x = 4 + y - 2z

36. x = 4 + (-1) - 2(-2/5)
37. x = 4 - 1 + 4/5
38. x = 3 + 4/5
39. x = 15/5 + 4/5
40. x = 19/5

Therefore, the solution is x = 19/5, y = -1, and z = -2/5.

Problem 7: Solve the system of equations using the elimination method.

x + 2y - z = 3

2x - y + z = 2

x + y + z = 4

Solution:

1. Add the first and second equations:
2. x + 2y - z = 3
3. 2x - y + z = 2
4. ---------------
5. 3x + y = 5
6. Subtract the first equation from the third:
7. x + y + z = 4
8. x + 2y - z = 3
9. ---------------
10. -y + 2z = 1
11. Now we have:
12. 3x + y = 5

13. -y + 2z = 1

14. Add these two equations:

15. 3x + y = 5

16. -y + 2z = 1

17. ---------------

18. 3x + 2z = 6

19. From the original third equation:

20. x + y + z = 4

21. Multiply by 3:

22. 3x + 3y + 3z = 12

23. Subtract the new equation from the previous one:

24. 3x + 2z = 6

25. 3x + 3y + 3z = 12

26. ---------------

27. -3y - z = -6

28. Solve for y in terms of z:

29. -3y - z = -6

30. -3y = -6 + z

31. y = 2 - z/3

32. Substitute into the equation -y + 2z = 1:

33. -(2 - z/3) + 2z = 1

34. -2 + z/3 + 2z = 1

35. -2 + z/3 + 6z/3 = 1

36. -2 + 7z/3 = 1

37. 7z/3 = 3
38. z = 9/7
39. Back-substitute to find y:
40. y = 2 - z/3
41. y = 2 - (9/7)/3
42. y = 2 - 9/21
43. y = 2 - 3/7
44. y = 14/7 - 3/7
45. y = 11/7
46. Back-substitute to find x:
47. x + y + z = 4
48. x + 11/7 + 9/7 = 4
49. x + 20/7 = 4
50. x = 4 - 20/7
51. x = 28/7 - 20/7
52. x = 8/7

Therefore, the solution is x = 8/7, y = 11/7, and z = 9/7.

Problem 8: Solve the system of equations using the substitution method.

3x + 2y = 1

4x - 5y = 17

Solution:

1. From the first equation:
2. 3x + 2y = 1

3. 2y = 1 - 3x
4. y = (1 - 3x)/2
5. Substitute into the second equation:
6. 4x - 5((1 - 3x)/2) = 17
7. 4x - 5(1 - 3x)/2 = 17
8. 4x - 5/2 + 15x/2 = 17
9. 4x + 15x/2 - 5/2 = 17
10. 8x/2 + 15x/2 - 5/2 = 17
11. 23x/2 - 5/2 = 17
12. 23x/2 = 17 + 5/2
13. 23x/2 = 34/2 + 5/2
14. 23x/2 = 39/2
15. 23x = 39
16. x = 39/23
17. Back-substitute to find y:
18. y = (1 - 3x)/2
19. y = (1 - 3(39/23))/2
20. y = (1 - 117/23)/2
21. y = (23/23 - 117/23)/2
22. y = (-94/23)/2
23. y = -47/23

Therefore, the solution is x = 39/23 and y = -47/23.

Problem 9: Solve the system of equations using the elimination method.

$x + y + z = 9$

$2x - y + z = 8$

$x + 2y - z = 2$

Solution:

1. Subtract the first equation from the second:

   $x + y + z = 9$

   $2x - y + z = 8$

   $x - 2y = -1$

2. Subtract the first equation from the third:

   $x + y + z = 9$

   $x + 2y - z = 2$

   $y - 2z = -7$

3. From the second equation:

   $y - 2z = -7$

   $y = -7 + 2z$

4. Substitute into the first equation:

   $x - 2y = -1$

   $x - 2(-7 + 2z) = -1$

   $x + 14 - 4z = -1$

   $x - 4z = -15$

5. From the original first equation:

   $x + y + z = 9$

6. Substitute the expression for y:

   x + (-7 + 2z) + z = 9

   x - 7 + 2z + z = 9

   x + 3z = 16

7. Now we have:

   x - 4z = -15

   x + 3z = 16

8. Subtract the first from the second:

   7z = 31

   z = 31/7

9. Back-substitute to find x:

   x + 3z = 16

   x + 3(31/7) = 16

   x + 93/7 = 16

   x = 16 - 93/7

   x = 112/7 - 93/7

   x = 19/7

10. Back-substitute to find y:

   y = -7 + 2z

   y = -7 + 2(31/7)

   y = -7 + 62/7

   y = -49/7 + 62/7

   y = 13/7

Therefore, the solution is x = 19/7, y = 13/7, and z = 31/7.

Problem 10: Solve the system of equations using the substitution method.

x + 2y - 3z = -3

2x - y + z = 7

-x + 3y - z = -6

Solution:

1. From the first equation:

   x + 2y - 3z = -3

   x = -3 - 2y + 3z

2. Substitute into the second equation:

   2(-3 - 2y + 3z) - y + z = 7

   -6 - 4y + 6z - y + z = 7

   -6 - 5y + 7z = 7

   -5y + 7z = 13

3. Substitute into the third equation:

   -(-3 - 2y + 3z) + 3y - z = -6

   3 + 2y - 3z + 3y - z = -6

   3 + 5y - 4z = -6

   5y - 4z = -9

   Now we have:

   -5y + 7z = 13

   5y - 4z = -9

4. Add these equations:

-5y + 7z = 13

5y - 4z = -9

3z = 4

z = 4/3

5. Back-substitute to find y:

5y - 4z = -9

5y - 4(4/3) = -9

5y - 16/3 = -9

5y = -9 + 16/3

5y = -27/3 + 16/3

5y = -11/3

y = -11/15

6. Back-substitute to find x:

x = -3 - 2y + 3z

x = -3 - 2(-11/15) + 3(4/3)

x = -3 + 22/15 + 12/3

x = -3 + 22/15 + 60/15

x = -3 + (22 + 60)/15

x = -3 + 82/15

x = -45/15 + 82/15

x = 37/15

Therefore, the solution is x = 37/15, y = -11/15, and z = 4/3.

Problem 11: Solve the system of equations using the elimination method.

2x + 3y = 8

4x - 3y = 1

Solution:

1. Multiply the first equation by 2:

   4x + 6y = 16

2. Add this to the second equation:

   4x + 6y = 16

   4x - 3y = 1

   9y = 17

   Solve for y:

   y = 17/9

3. Substitute back into the first equation:

   2x + 3(17/9) = 8

   2x + 51/9 = 8

   2x = 8 - 51/9

   2x = 72/9 - 51/9

   2x = 21/9

   x = 21/18

   x = 7/6

Therefore, the solution is x = 7/6 and y = 17/9.

Problem 12: Solve the system of equations using the substitution method.

x + y = 5

2x - 3y = -4

Solution:

1. From the first equation:

   x + y = 5

   x = 5 - y

2. Substitute into the second equation:

   2(5 - y) - 3y = -4

   10 - 2y - 3y = -4

   10 - 5y = -4

   -5y = -14

   y = 14/5

3. Back-substitute to find x:

   x = 5 - y

   x = 5 - 14/5

   x = 25/5 - 14/5

   x = 11/5

Therefore, the solution is x = 11/5 and y = 14/5.

Problem 13: Solve the system of equations using the elimination method.

x + y + z = 6

2x + y - z = 3

x - y + 2z = 4

Solution:

1. Subtract the first equation from the second:

x + y + z = 6

2x + y - z = 3

----

x - 2z = -3

2. Subtract the first equation from the third:

   x + y + z = 6

   x - y + 2z = 4

   -2y + z = -2

3. Multiply the second equation by 2:

   2x - 4z = -6

4. Add this to the first equation:

   x - 2z = -3

   2x - 4z = -6

   3x - 6z = -9

5. Simplify:

   3x - 6z = -9

   x - 2z = -3

6. From the second equation:

   -2y + z = -2

   z = -2 + 2y

7. Substitute into the first equation:

x - 2z = -3

x - 2(-2 + 2y) = -3

x + 4 - 4y = -3

x - 4y = -7

8. From the original first equation:

x + y + z = 6

9. Substitute the expression for z:

x + y + (-2 + 2y) = 6

x + y - 2 + 2y = 6

x + 3y = 8

Now we have:

x - 4y = -7

x + 3y = 8

10. Subtract the first from the second:

7y = 15

y = 15/7

11. Back-substitute to find x:

x + 3y = 8

x + 3(15/7) = 8

x + 45/7 = 8

x = 8 - 45/7

x = 56/7 - 45/7

x = 11/7

12. Back-substitute to find z:

z = -2 + 2y

z = -2 + 2(15/7)

z = -2 + 30/7

z = -14/7 + 30/7

z = 16/7

Therefore, the solution is x = 11/7, y = 15/7, and z = 16/7.

Problem 14: Solve the system of equations using the substitution method.

3x - 2y = 7

2x + 5y = 1

Solution:

1. From the first equation:

   3x - 2y = 7

   3x = 7 + 2y

   x = (7 + 2y)/3

2. Substitute into the second equation:

   2((7 + 2y)/3) + 5y = 1

   2(7 + 2y)/3 + 5y = 1

   14/3 + 4y/3 + 5y = 1

   14/3 + 4y/3 + 15y/3 = 1

   14/3 + 19y/3 = 1

   14 + 19y

Multiple Choice Questions (MCQs)

1. The determinant of a 2×2 matrix [abcd]\begin{bmatrix} a & b \\ c & d \end{bmatrix}[acbd] is given by:

a) a+da + da+d

b) a×d−b×ca \times d - b \times ca×d−b×c

c) a×b−c×da \times b - c \times da×b−c×d

d) a−da - da−d

2. If a matrix has the same number of rows and columns, it is called:

   a) Rectangular Matrix

   b) Square Matrix

   c) Identity Matrix

   d) None of the above

3. The inverse of a matrix exists if and only if its determinant is:

   a) Zero

   b) Positive

   c) Non-zero

   d) Negative

4. What is the result of multiplying a matrix by a scalar?

   a) The matrix remains unchanged

   b) Each element of the matrix is multiplied by the scalar

   c) Only the diagonal elements change

   d) The determinant changes but the elements remain the same

5. Which of the following is NOT a type of matrix?

   a) Identity Matrix

   b) Square Matrix

   c) Rectangular Matrix

   d) Inverse Matrix

6. The determinant of an identity matrix is always:

   a) 0

   b) 1

   c) -1

   d) Depends on the order of the matrix

7. The inverse of a matrix A is denoted by:

a) A−1A^{-1}A−1

b) A′A'A′

c) ATA^TAT

d) |A||A||A|

8. If two matrices A and B are multiplied, then the resulting matrix is obtained by:

   a) Adding corresponding elements

   b) Subtracting corresponding elements

   c) Row by column multiplication rule

   d) Multiplying corresponding elements

9. Which of the following conditions must be satisfied for matrix multiplication to be possible?

   a) Number of rows of first matrix = Number of columns of second matrix

   b) Both matrices must be square matrices

   c) Determinants of both matrices must be equal

   d) Both matrices must be of the same order

10. If the determinant of a matrix is 0, the matrix is called:

    a) Singular Matrix

    b) Non-Singular Matrix

    c) Diagonal Matrix

    d) Unit Matrix

Short Answer Questions

1. Define determinant and mention its properties.
2. What are the different types of matrices? Explain with examples.
3. Explain how to find the inverse of a matrix using its adjoint.
4. Describe the steps to solve a system of linear equations using matrices.
5. State and explain the conditions for matrix multiplication to be possible.

MODULE III SIMPLE INTEREST, COMPOUND INTEREST, AND PROFIT & LOSS

Structure

3.0 OBJECTIVES

To understand the concepts of simple and compound interest.

To explore the present value and amount of annuities.

To analyze the principles of profit and loss in business transactions**.**

UNIT 8 Simple Interest (SI)

Simple Interest: Simple interest is an interest calculation based on the principal, interest rate, and the amount of time that passes. The interest is not added to the principal for subsequent interest calculations, unlike compound interest.Simple interest can be calculated using this formula:

$SI = P \times r \times t$

Where:

• SI = Simple Interest

P = Principal (the initial amount invested or borrowed)

• r = Rate of interest (in decimal value like 5% = 0.05)

• t = Time period (typically years)

$A = P(1 + rt)$ $A = P + Prt$, where P & r is the principal amount and interest rate respectively.

$A = P + SI = P + (P \times r \times t) = P(1 + r \times t)$

Step 1: Set up equations

From the first condition (interest earned in one year is $310), we can write:

$$0.05x + 0.07y = 310$$

From the second condition (interest earned would be $10 more if the amounts were flipped), we write:

$$0.05y + 0.07x = 320$$

Step 2: Subtract equation (1) from equation (2)

Subtract equation (1) from equation (2) to eliminate the interest amounts:

$$(0.05y + 0.07x) - (0.05x + 0.07y) = 320 - 310$$

Simplifying this:

$$0.05y + 0.07x - 0.05x - 0.07y = 10$$

$$0.02x - 0.02y = 10$$

Dividing both sides by 0.02:

$$y = x - 500$$

Step 3: Substitute equation (3) into equation (1)

From equation (3), we know that:

$$y = x - 500$$

Now, substitute this into equation (1):

$$0.05x + 0.07(x - 500) = 310$$

Simplifying:

$$0.05x + 0.07x - 0.07 \times 500 = 310$$

$$0.12x - 35 = 310$$

$$0.12x = 310 + 35$$

$$0.12x = 345$$

Solving for xxx:

$$x = \frac{345}{0.12} = 2875$$

Step 4: Calculate yyy

Now that we know x=2875x = 2875x=2875, substitute this back into equation (3) to find yyy:

$$y = x - 500 = 2875 - 500 = 2375$$

Final Answer:

- Mrs. Garcia invested $2,875 at 5% per year.
- Mrs. Garcia invested $2,375 at 7% per year.

Step 1: Calculate the Interest for Two Years

The formula for simple interest is:

$$\text{Interest} = P \times r \times t$$

Substitute the known values:

$$\text{Interest} = 10,000 \times 0.06 \times 2 = 1,200$$

tep 2: Calculate the Total Amount to Pay Back

The total amount Mr. Williams has to pay back is the principal plus the interest:

$$\text{Total to pay back} = P + \text{Interest} = 10,000 + 1,200 = 11,200$$

Step 3: Calculate Each Instalment

Since Mr. Williams is making two equal payments, each payment is:

$$\text{Each instalment} = \frac{11,200}{2} = 5,600$$

Final Answer:

The amount of each installment is **$5,600**.

Problem 20: A sum of money at simple interest amounts to 2.5 times itself in 15 years. What is the rate of interest?

Solution:

Let the principal be ***P***. After 15 years, the amount becomes ***2.5P.***

We use the formula for simple interest:

$$A = P(1 + rt)$$

Where:

- A = Amount after time t
- P = Principal
- r = Rate of interest per annum (in decimal form)
- t = Time (in years)

Substitute the values into the formula:

$$2.5P = P(1 + r \times 15)$$

Now, divide both sides by *P:*

$$2.5 = 1 + 15r$$

Solve for *r:*

$$2.5 - 1 = 15r$$

$$1.5 = 15r$$

$$r = \frac{1.5}{15} = 0.1$$

Thus, the rate of interest is:

$$r = 0.1 \quad \text{or} \quad r = 10\% \text{ per annum.}$$

Final Answer:

The interest rate is **10% per annum**.

Problem 21: The simple interest on a certain principal at the rate of 7% per annum for 3 years is $420. How much will the amount be in 5 years?

Solution:

Let *P* be the principal.

Step 1: Find the Principal

We are given the simple interest (SI) for 3 years as $420, with an interest rate of 7% per annum. We can use the simple interest formula:

$$\text{SI} = P \times r \times t$$

Substitute the given values:

$$420 = P \times 0.07 \times 3$$

Solve for ***P:***

$$420 = P \times 0.21$$

$$P = \frac{420}{0.21} = 2,000$$

So, the principal is **$2,000**.

Step 2: Calculate the Simple Interest for 5 Years

Now, we calculate the simple interest for 5 years using the principal P=2,000 the interest rate r=7%=0.07, and the time t =5 years:

$$\text{SI for 5 years} = P \times r \times t = 2,000 \times 0.07 \times 5 = 700$$

So, the interest for 5 years is **$700**.

Step 3: Calculate the Total Amount

The total amount after 5 years is the sum of the principal and the interest:

$$\text{Amount} = P + \text{SI} = 2,000 + 700 = 2,700$$

Final Answer*:*

After 5 years, the total amount will be **$2,700**.

Problem 22: Mr. Thompson takes out a loan of $5,000 at a simple interest rate of 9% per annum. How much more does he need to pay at the end of the second year if he repays $2,000 at the end of the first year, in order to discharge his debt at the end of the second year?

Solution:

Given:

- Principal P=5,000
- Interest rate r=9%=0.09
- Time t=2 years

Step 1: Calculate the Simple Interest

The formula for simple interest is:

$$\text{Simple Interest}(SI) = P \times r \times t$$

Substituting the known values:

$$SI = 5,000 \times 0.09 \times 2 = 900$$

So, the simple interest for 2 years is **$900**.

Step 2: Calculate the Total Amount to Be Repaid After 2 Years

The total amount to be repaid after 2 years is the sum of the principal and the interest:

$$SI = 5,000 \times 0.09 \times 2 = 900$$

So, the total amount to be repaid after 2 years is **$5,900**.

Step 3: Subtract the Repayment at the End of the First Year

Mr. Thompson repays $2,000 at the end of the first year. To determine how much more he needs to pay at the end of the second year:

$$\text{Total Amount} = P + SI = 5,000 + 900 = 5,900$$

Thus, the amount Mr. Thompson needs to pay at the end of the second year is **$3,900**.

$$0.05x + 0.08y = 580 \quad \text{(Equation 2)}$$

Step 1: Solve for yyy in terms of xxx from Equation (1)

From Equation (1), we can express yyy as:

$$y = 9,000 - x$$

Step 2: Substitute yyy into Equation (2)

Now, substitute y=9,000−x into Equation (2):

Step 3: Solve for x

Now, solve for x

$$-0.03x = 580 - 720$$

$$-0.03x = -140$$

$$x = \frac{-140}{-0.03} = 4,666.67$$

So, the amount invested at 5% is **$4,666.67**.

Step 4: Find y

Using Equation (1), substitute x=4,666.67x to find y:

$$y = 9,000 - 4,666.67 = 4,333.33$$

So, the amount invested at 8% is **$4,333.33**.

Final Answer:

The man invested **$4,666.67** at 5% interest and **$4,333.33** at 8% interest.

Problem 24: A certain amount of money becomes double in 10 years at simple interest. How many years will it take for it to become three times larger at the same interest rate?

Solution:

Let:

- P be the principal
- r be the rate of interest

Step 1: Calculate the Interest Rate

To double the principal in 10 years, we use the simple interest formula:

$$A = P(1 + rt)$$

For doubling the principal, the amount becomes 2P2P2P after 10 years, so:

$$2P = P(1 + r \times 10)$$

Simplify:

$$2P = P(1 + 10r)$$

Divide both sides by P:

$$2 = 1 + 10r$$

Solve for r:

$$2 - 1 = 10r$$

$$r = \frac{1}{10} = 0.1 = 10\%$$

So, the rate of interest is **10% per annum**.

Step 2: Calculate the Time for the Amount to Triple

Now, we want to find the time ttt for the principal to become three times its original amount. Using the same formula for simple interest, we have:

$$A = P(1 + rt)$$

For tripling the principal, the amount becomes 3P3P3P, so:

$$3P = P(1 + 0.1 \times t$$

Simplify:

$$3P = P(1 + 0.1t)$$

Divide both sides by *P:*

$$3 = 1 + 0.1t$$

Solve for *t*:

$$3 - 1 = 0.1t$$

$$2 = 0.1t$$

Step 1: Calculate the Interest for 3 Years

Using the simple interest formula:

$$\text{Interest} = P \times r \times t$$

Substitute the given values:

$$\text{Interest} = 12,000 \times 0.075 \times 3 = 2,700$$

So, the total interest for 3 years is **$2,700**.

Step 2: Calculate the Total Amount to Be Paid

The total amount to be repaid is the principal plus the interest:

$$\text{Total Amount} = P + \text{Interest} = 12,000 + 2,700 = 14,700$$

Step 3: Calculate the Annual Installment

Since the loan is paid off in 3 equal installments, the amount of each installment is:

$$\text{Annual Installment} = \frac{14,700}{3} = 4,900$$

Final Answer:

The amount of each annual installment is **$4,900**.

Problem 26: A certain sum of money at simple interest amounts to $1,320 in 2 years and $1,380 in 3 years.

Solution:

Let the principal be PPP and the rate of interest be *r*.

Step 1: Set up the equations based on simple interest

From the information given:

- After 2 years, the amount is $1,320:

$$P + P \times r \times 2 = 1,320 \quad \Rightarrow \quad P(1 + 2r) = 1,320 \quad (\text{Equation 1})$$

- After 3 years, the amount is $1,380:

$$P + P \times r \times 3 = 1,380 \quad \Rightarrow \quad P(1 + 3r) = 1,380 \quad (\text{Equation 2})$$

Step 2: Subtract Equation 1 from Equation 2

Subtract Equation 1 from Equation 2 to eliminate *P:*

$$P(1+3r) - P(1+2r) = 1,380 - 1,320$$

Simplifying:

$$P(3r - 2r) = 60$$

$$P \times r = 60 \quad \text{(Equation 3)}$$

Step 3: Substitute Equation 3 into Equation 1

Now, substitute P×r=60P \times r = 60P×r=60 into Equation 1:

$$P + 2 \times 60 = 1,320$$

$$P + 120 = 1,320$$

$$P = 1,320 - 120 = 1,200$$

Thus, the principal is **$1,200**.

Step 4: Calculate the rate of interest

Using Equation 3:

$$P \times r = 60$$

$$1,200 \times r = 60$$

$$r = \frac{60}{1,200} = 0.05 = 5\%$$

Thus, the rate of interest is **5% per annum**.

Problem 27: Mrs. Wilson invested a certain amount at a simple interest rate of 6%. She received $1,440 in interest. For how many years was the money invested?

Solution:

Let:

- Principal P = 8,000
- Rate of interest r= 6% = 0.06
- Simple Interest SI=1,440

Step 1: Use the Simple Interest Formula

The formula for simple interest is:

$$\text{SI} = P \times r \times t$$

Substitute the known values:

$$1,440 = 8,000 \times 0.06 \times t$$

Step 2: Solve for ttt

Simplify the equation:

$$1,440 = 480 \times t$$

$$t = \frac{1,440}{480} = 3 \text{ years}$$

Thus, the investment time is **3 years**.

$$0.06x + 0.09y = 780 \quad (\text{Equation } 2)$$

Step 1: Solve the system of equations

From Equation 1, we can express yyy in terms of *x:*

$$y = 10,000 - x$$

Substitute this expression for yyy into Equation 2:

$$0.06x + 0.09(10,000 - x) = 780$$

Simplify the equation:

$$0.06x + 900 - 0.09x = 780$$

Combine like terms:

Subtract 900 from both sides:

$$-0.03x = -120$$

Solve for xxx:

$$x = \frac{-120}{-0.03} = 4,000$$

So, the first person invested **$4,000**.

Step 2: Calculate the amount for the second person

From Equation 1:

$$y = 10,000 - 4,000 = 6,000$$

So, the second person invested **$6,000**.

Step 3: Verify the solution

We can verify the interest earned using Equation 2:

$$0.06 \times 4,000 + 0.09 \times 6,000 = 240 + 540 = 780$$

The total interest is indeed $780, confirming that the first person invested $4,000 and the second person invested $6,000.

Problem 29: A certain sum amounts to $5,500 in 2 years and $6,500 in 4 years at simple interest. Find the principal and the rate of interest.

Solution:

Let:

- P be the principal.
- r be the rate of interest.

Step 1: Set up the equations

From the information given:

- After 2 years, the amount is $5,500:

$$P + P \times r \times 2 = 5,500 \quad \Rightarrow \quad P(1+2r) = 5,500 \quad \text{(Equation 1)}$$

- After 4 years, the amount is $6,500:

$$P + P \times r \times 4 = 6,500 \quad \Rightarrow \quad P(1+4r) = 6,500 \quad \text{(Equation 2)}$$

Step 2: Subtract Equation 1 from Equation 2

Subtract Equation 1 from Equation 2 to eliminate PPP:

$$P(1+4r) - P(1+2r) = 6,500 - 5,500$$

Simplify:

$$P(4r - 2r) = 1,000$$

$$2Pr = 1,000$$

$$Pr = 500 \quad \text{(Equation 3)}$$

Step 3: Solve for the principal

Substitute Pr=500Pr = 500Pr=500 into Equation 1:

$$P + 2 \times 500 = 5,500$$

$$P + 1,000 = 5,500$$

$$P = 5,500 - 1,000 = 4,500$$

Thus, the principal is **$4,500**.

Step 4: Solve for the rate of interest

From Equation 3:

$$4,500 \times r = 500$$

$$r = \frac{500}{4,500} = 0.111 = 11.1\%$$

Thus, the rate of interest is **11.1% per annum**.

Further Applications of Simple Interest

Commercial and Business Applications

1. Business Loans

Simple interest is commonly used for short-term business loans, especially working capital financing.

For example, a company takes a working capital loan of $50,000 at 7% simple interest for 6 months. The interest would be: SI = $50,000 × 0.07 × (6/12) = $1,750 Total repayment = $50,000 + $1,750 = $51,750

Trade Credit

Suppliers typically add simple interest to late payments.

For instance, suppliers may offer "3/10, net 30" terms, which means that a buyer can have a discount of 3% if the supplier is paid within 10 days,

UNIT 9 Compound Interest and Annuities

Compound interest refers to the interest that is calculated on both the initial principal as well as the interest on the previous periods of deposits or loans. In broad terms, it means compound interest – interest paid on an interest on an interest, and one of the most powerful financial concepts that leads to a compound growth of money over time.This is incredibly important to understand, because unlike simple interest, which is only calculated on the original principal, the magic of compound interest creates a snowball effect, exponentially increasing your investment or debt growth the longer you hold onto it over time.

Basic Required Formula For Compound Interest: The formula when interest is compounded periodically is:-

A = P(1 + r/n)^(nt)

Where:

- A = Final amount (principal + interest)
- P = Principal (initial investment or loan amount)
- r = Annual interest rate (as a decimal)
- n = Number of compounding periods (i.e. Interest Compounding Frequency) per year
- t = Time in years

If, instead, the interest compounds annually, the formula simplifies to:

A = P(1 + r)^t

Compound interest itself is calculated by:

CI = A - P

Calculation of CI

1. Calculating compound interest involves a few straightforward steps:
2. Example (1 rate 2 rate 3 rate): pp for variable interest steerer (APY)

3. Find out how often interest is compounded during a year

4. The final amount can be calculated using the formula
5. And subtract the initial principal to calculate the compound interest earned

Example 1: Calculate the compound interest on $5,000 invested for 3 years at an interest rate of 6% per annum, compounded annually.

Solution:

Given:

- Principal (*P)* = $5,000
- Rate of interest (*r*) = 6% = 0.06
- Time (*t*) = 3 years
- Compounding frequency (*n)* = 1 (annually)

Using the compound interest formula:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Substitute the known values into the formula:

$$A = 5,000\left(1 + \frac{0.06}{1}\right)^{1\times3}$$

$$A = 5,000\,(1.06)^3$$

$$A = 5,000 \times 1.191016$$

$$A = 5,955.08$$

The total amount after 3 years is **$5,955.08**.

Step 2: Calculate the Compound Interest

The compound interest is the difference between the total amount and the principal:

$$\text{Compound Interest} = A - P = 5,955.08 - 5,000 = 955.08$$

Thus, the compound interest earned is **$955.08**.

Final Answer:

The compound interest on $5,000 invested for 3 years at 6% per annum, compounded annually, is **$955.08**.

Example 2: Calculate the compound interest on $10,000 invested for 2 years at 8% per annum, compounded quarterly.

Solution:

- P = $10,000
- r = 8% = 0.08
- t = 2 years
- n = 4 (quarterly)

Using the formula: A = P(1 + r/n)^(nt) A = 10000(1 + 0.08/4)^(4×2) A = 10000(1.02)^8 A = 10000 × 1.1716842 A = $11,716.84

Compound Interest = A - P = $11,716.84 - $10,000 = $1,716.84

Comparison between SI and CI: Simple Interest (SI) and Compound Interest (CI) represent two fundamentally different approaches to calculating interest.

Key Differences:

1. **Calculation Basis:**

   - Simple Interest: Calculated only on the principal amount

- Compound Interest: Calculated on the principal and previously accumulated interest

2. **Growth Pattern:**

- Simple Interest: Linear growth (constant amount of interest each period)
- Compound Interest: Exponential growth (accelerating amount of interest each period)

3. **Formula:**

- Simple Interest: $A = P(1 + rt)$
- Compound Interest: $A = P(1 + r/n)^{(nt)}$

4. **Effect of Time:**

- As time increases, the difference between SI and CI becomes more significant

5. **Compounding Frequency:**

- More frequent compounding in CI (monthly vs. annually) results in higher returns

Numerical Comparison Example: Let's compare SI and CI for an investment of \$10,000 at 5% annual interest rate over 10 years:-

Simple Interest calculation: $SI = P \times r \times t$ $SI = \$10,000 \times 0.05 \times 10$ $SI = \$5,000$

Total amount with SI = \$10,000 + \$5,000 = \$15,000

Compound Interest calculation (annual compounding): $A = P(1 + r)^t$ $A = \$10,000(1 + 0.05)^{10}$ $A = \$10,000 \times 1.6289$ $A = \$16,289$

CI = \$16,289 - \$10,000 = \$6,289

Difference between CI and SI = \$6,289 - \$5,000 = \$1,289

This example demonstrates that over a 10-year period, compound interest generates $1,289 more than simple interest on the same principal.

The Rule of 72:

A useful shortcut to estimate how long it takes for money to double with compound interest is the "Rule of 72":

Years to double = 72 ÷ Annual interest rate (%)

For example, at 6% compound interest, money doubles in approximately 72 ÷ 6 = 12 years.

UNIT 10 Present Value and Annuity

Concept of Annuity

An annuity is a series of equal payments made at equal time intervals. Annuities are common in various financial arrangements:

- Regular deposits into savings accounts
- Mortgage payments
- Insurance premium payments
- Retirement pension distributions
- Loan repayments

Types of Annuities:

1. **Ordinary Annuity (Annuity in Arrears):**
   - Payments occur at the end of each period
   - Common in loan repayments
2. **Annuity Due:**
   - Payments occur at the beginning of each period
   - Common in lease payments and insurance premiums
3. **Perpetuity:**

- An annuity that continues indefinitely
- Has no end date

4. **Fixed Annuity:**
   - Payment amounts remain constant
5. **Variable Annuity:**
   - Payment amounts may change over time

For our discussion, we'll focus primarily on ordinary annuities with fixed payment amounts.

Present Value of Annuity

The present value of an annuity is the current worth of a series of future equal payments, discounted at a specified interest rate. It represents how much money would need to be invested now to generate the future payment stream.

This concept is essential for:

- Valuing pension benefits
- Determining fair prices for financial products
- Calculating loan payoff amounts
- Planning retirement savings

Present Value of an Ordinary Annuity Formula:

$PV = PMT \times [(1 - (1 + r)^{(-n)}) / r]$

Where:

- PV = Present value
- PMT = Payment amount per period
- r = Interest rate per period
- n = Total number of payments

Example 1 Calculate the present value of an ordinary annuity of $1,000 paid annually for 5 years, with an interest rate of 6% per annum.

Solution:

Given:

- Payment per period (PMT) = $1,000
- Interest rate (r) = 6% = 0.06
- Number of periods (n) = 5 years

We will use the formula for the present value of an ordinary annuity:

$$PV = PMT \times \left[\frac{1-(1+r)^{-n}}{r}\right]$$

Substituting the known values into the formula:

$$PV = 1,000 \times \left[\frac{1-(1+0.06)^{-5}}{0.06}\right]$$

Simplifying the terms inside the brackets:

$$PV = 1,000 \times \left[\frac{1-(1.06)^{-5}}{0.06}\right]$$

Calculating (1.06)−5:

$$(1.06)^{-5} \approx 0.7473$$

Now, substitute this value back:

$$PV = 1,000 \times \left[\frac{1 - 0.7473}{0.06}\right]$$

$$PV = 1,000 \times \left[\frac{0.2527}{0.06}\right]$$

$$PV = 1,000 \times 4.2117$$

$$PV = 4,211.70$$

Final Answer:

The present value of receiving $1,000 annually for 5 years with a 6% interest rate is **$4,211.70**.

This means that receiving $1,000 annually for 5 years at a 6% interest rate is equivalent to having $4,211.70 today.

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Example 2: Calculate the present value of $500 monthly payments for 3 years if money is worth 8% per year compounded monthly.

Solution:

Given:

- Monthly payment (*PMT*) = $500
- Annual interest rate (*r*) = 8% per year = 0.08
- Monthly interest rate = 0.0812 = 0.0067 = (since the interest is compounded monthly)
- Number of periods (nnn) = 3 years × 12 months = 36 months

We will use the formula for the present value of an ordinary annuity:

$$PV = PMT \times \left[\frac{1 - (1 + r)^{-n}}{r}\right]$$

Substituting the known values into the formula:

$$PV = 500 \times \left[\frac{1-(1+0.0067)^{-36}}{0.0067}\right]$$

Simplifying inside the brackets:

$$PV = 500 \times \left[\frac{1-(1.0067)^{-36}}{0.0067}\right]$$

Calculating (1.0067)−36(1.0067)^{-36}(1.0067)−36:

$$(1.0067)^{-36} \approx 0.7857$$

Substitute this back:

$$PV = 500 \times \left[\frac{1-0.7857}{0.0067}\right]$$

$$PV = 500 \times \left[\frac{0.2143}{0.0067}\right]$$

$$PV = 500 \times 31.9851$$

$$PV = 15,992.55$$

Final Answer:

The present value of receiving $500 monthly payments for 3 years with an 8% annual interest rate compounded monthly is **$15,992.55**.

Future Value of an Annuity (Amount of Annuity)

The future value of an annuity is the total amount of all payments plus the interest accumulated by a specific future date. This is especially useful in savings and investment planning.

Formula for the Future Value of an Ordinary Annuity:

$$A = PMT \times \left[\frac{(1+r)^n - 1}{r}\right]$$

Where:

- A = Future value (amount)
- PMT = Payment amount per period
- r = Interest rate per period
- n = Total number of payments/periods

This formula helps calculate how much the annuity will accumulate over time based on the specified payments and interest rate.

Example 1: Calculate the amount of an ordinary annuity if $2,000 is deposited at the end of each year for 6 years in an account paying 5% interest compounded annually.

Solution:

Given:

- Payment per period (*PMT)* = $2,000
- Interest rate *(r)* = 5% = 0.05
- Number of periods (*n*) = 6 years

Using the formula for the future value of an ordinary annuity:

$$A = PMT \times \left[\frac{(1+r)^n - 1}{r}\right]$$

Substitute the known values:

$$A = 2000 \times \left[\frac{(1 + 0.05)^6 - 1}{0.05} \right]$$

First, calculate (1+0.05)6(1 + 0.05)^6(1+0.05)6:

$$(1 + 0.05)^6 = 1.3401$$

Now, substitute this back into the formula:

$$A = 2000 \times \left[\frac{1.3401 - 1}{0.05} \right]$$

$$A = 2000 \times \left[\frac{0.3401}{0.05} \right]$$

$$A = 2000 \times 6.802$$

$$A = 13,604$$

Thus, after 6 years, the annuity will accumulate to **$13,604**.

Example 2: Problem: John deposits $300 at the end of each month into a savings account that pays 6% per annum compounded monthly. How much will he have accumulated after 4 years?

Solution:

Given:

- Payment per period *(PMT)* = $300
- Annual interest rate (*r*) = 6% per year = 6%12=0.5%=0.005 per month
- Number of periods (*n)* = 4 years × 12 months = 48 months

Using the formula for the future value of an ordinary annuity:

$$A = PMT \times \left[\frac{(1 + r)^n - 1}{r} \right]$$

Substitute the known values:

$$A = 300 \times \left[\frac{(1+0.005)^{48} - 1}{0.005}\right]$$

First, calculate (1+0.005)48(1 + 0.005)^{48}(1+0.005)48:

$$(1+0.005)^{48} = 1.2705$$

Now, substitute this back into the formula:

$$A = 300 \times \left[\frac{1.2705 - 1}{0.005}\right]$$

$$A = 300 \times \left[\frac{0.2705}{0.005}\right]$$

$$A = 300 \times 54.1$$

$$A = 16,230$$

Thus, after 4 years of monthly deposits, John will have accumulated **$16,230**.

UNIT 11Profit and Loss

Cost Price (CP)

Cost that a seller pays to purchase or produce an price is the amount of money asset. In addition to the original purchase price, this accounts for any additional costs involved in preparing those items for sale, like expenses, and packaging. transportation, storage, labor costs, manufacturing total amount of the investment a business makes in a The cost price is the product before it's introduced to customers.

Selling Price (SP)

The money a customer must pay to purchase a selling price is the total amount of ultimately pays for the good product or service. That’s the price the consumer fixed, the selling price is left to the in question. Although the cost price is discretion of the seller, and is typically set to ensure recovery of cost price is less than cost along with a profit margin. But in some cases selling price price so loss occurs.

Marked Price (MP)

The price or maximum retail price (MRP) is the price that is marked price, list printed on the product before any discounts or offers are applied on it. The selling price is usually higher than the cost price and the selling price the can make an offer while still business intends to sell the product at, so they idea of saving and value they get maintaining profit. This gives customers an when they have discounts on the marked price.

Profit and Loss Formulas

Simple for Profit and Loss Calculations

Profit

Selling Price: The Selling Price (SP) of an item is the price at which the item is in the market, If SP sold > CP then seller makes a profit.

Profit • = Selling Price (SP) – Cost Price (CP)

• / Cost Price) × 100% Profit Percent = (Profit

Loss

Seller is less than the cost price. Loss: A situation in which the sale price

• − Selling Price (SP) Loss = Cost Price (CP)

• / Cost Price) × 100% Percentage Loss = (Loss

Calculating and Selling Price the Cost Price

Finding Cost Price

When CP = SP ÷ (1 + Profit% ÷ 100) profit is known:

If 100) loss is known: CP = SP ÷ (1 - Loss% ÷

Finding Selling Price

• / 100) When profit is given: SP = CP × (1 + Profit%

= CP × (1 - Loss% ÷ 100) When loss is known: SP

_uccessive Transactions

Therefore, with the same item: when multiple transactions happen

Net after two consecutive transactions for profit profit/loss percentage percentages P_1 and P_2: $P_1 + P_2 + (P_1 \times P_2 \div 100)$ %

Discount Calculations

• — Selling Price (SP) Discount = Marked Price (MP)

Discount% • = (Discount/Marked Price) × 100%

• Discount% / 100) Selling Price (SP) = Marked Price (MP) × (1 –

Discount and Selling Strategies

Types of Discounts

1. Cash discount: A customer pays the cash without using credit cards or any option. other
2. Volume when consumers buy products Discount: Similar to quantity discount, but applied in volume or bulk.
3. And, demand? also gives at seasonal low
4. Trade to wholesalers, retailers or for anyone in the distribution Discount: Given channel.
5. Loyalty for continued discount: This is offered to returning customers as a reward patronage.
6. Promotional promotional events or marketing Discount: These are offered at the time of campaigns.

Selling Strategies

1. Loss Leader Pricing: Operating at a loss on certain items to attract customers who are in store. will then be enticed to buy higher-margin products once they
2. Psychological . 99 or. 95 so they look much lower than they Pricing — Ending the price in of $10). really are (like $9.99 instead

3. Volume Pricing: Discounting purchases of multiple unit quantities (e.g., "2, Buy Get 1 Free" or "3 for $10").
4. It at a small discount for is grouping up several items and selling them bundled the total price compared to buying each one separately
5. Markup selling price. Pricing: Standard percentage added to cost price for
6. Value-Based — Pricing: Pricing that is based on the success value of the company to the customer, rather than on the internal cost of production
7. It new can be skimming: the practice of charging very high initial prices for the products and subsequently lowering them.
8. If for yes, backpackers and travel enthusiasts will simply switch to pay accommodation and scan through to catch up on cultural exercise.

Multiple Choice Questions (MCQs)

1. The formula for Simple Interest is:
   a) SI=P×R×TSI = P \times R \times TSI=P×R×T
   b) SI=P×R×T100SI = \frac{P \times R \times T}{100}SI=100P×R×T
   c) SI=P+R+T100SI = \frac{P + R + T}{100}SI=100P+R+T
   d) SI=P+R+TSI = P + R + TSI=P+R+T

2. If the principal amount is ₹5000, the rate of interest is 5% per annum, and the time is 2 years, what is the Simple Interest?
   a) ₹500
   b) ₹1000
   c) ₹250
   d) ₹750

3. The formula for Compound Interest when compounded annually is:
   a) A=P(1+R×T)A = P(1 + R \times T)A=P(1+R×T)
   b) A=P(1+R100)TA = P(1 + \frac{R}{100})^TA=P(1+100R)T
   c) A=P+R+TA = P + R + TA=P+R+T
   d) A=P(1−R100)TA = P(1 - \frac{R}{100})^TA=P(1−100R)T

4. If ₹10,000 is invested at a 10% annual interest rate, compounded annually for 2 years, what will be the final amount?

) ₹11,000

b) ₹12,100

c) ₹12,000

d) ₹10,500

5. The difference between Compound Interest and Simple Interest for one year on a principal of ₹5000 at an interest rate of 5% is:

a) ₹10

b) ₹12.50

c) ₹25

d) ₹0

6. The present value of an annuity is:

a) The future value of money received over time

b) The sum of all annuity payments

c) The total amount obtained at the end of the period

d) The value of a series of future payments at the present time

7. If a shopkeeper sells an item at a profit of 20%, the selling price is:

a) 80% of the cost price

b) 100% of the cost price

c) 120% of the cost price

d) 150% of the cost price

8. If a product is marked at ₹800 and a discount of 10% is given, what is the selling price?

a) ₹700

b) ₹720

c) ₹750

d) ₹780

9. A person buys an item for ₹600 and sells it at ₹750. What is the percentage profit?

a) 10%

b) 20%

c) 25%

d) 30%

10. If the cost price of an article is ₹500 and the selling price is ₹400, what is the loss percentage?

a) 10%

b) 15%

c) 20%

d) 25%

Short Answer Questions

1. Define Simple Interest and give its formula.
2. How is Compound Interest different from Simple Interest? Explain with an example.
3. What is the concept of present value of annuity?
4. Explain the terms Cost Price, Selling Price, and Profit with examples.
5. A shopkeeper sells a product at a 25% discount. If the marked price is ₹1000, what is the final selling price?

MODULE IV LINEAR PROGRAMMING AND TRANSPORTATION PROBLEM

Structure

OBJECTIVES

To understand the formulation of Linear Programming Problems (LPP).

To solve LPP using graphical methods.

To analyze different cases of LPP solutions

Unit 12Linear Programming Problem (LPP)

It then introduces linear programming (LP), a mathematical optimization approach to finding a maximum or minimum of a mathematical model whose constraints are formulated by linear relations. Linear programming is a special case included in mathematical programming or mathematical optimization.In a nutshell, a Linear Programming Problem (LPP) to maximize (or minimize) a linear function given linear constraints. The function to be maximized or minimized is called the objective function, while the limitations on the decision variables are expressed as linear inequalities or equations..

Importance:

Linear Programming has widespread applications in various fields:

1. **Business and Economics:** Different resource allocation, production planning, transportation problems, investment decisions, profit maximization.

2. **Manufacturing**: Planning production schedules while controlling costs and increasing throughput.
3. **Agriculture**: Optimise crop mix, feed formulation and farm management.

4. **Transportation and Logistics**: Transportation and Logistics Route planning, supply chain optimization and distribution network design
5. **Military Operations**: Section of JOE focused on Resource allocation, Deployment strategies and Logistics planning.
6. **Healthcare**: Staff scheduling, resource management, and treatment planning.
7. **Energy Sector**: Power generation scheduling, fuel allocation and distribution planning

Linear programming comes in handy when dealing with complex real-life problems which have many variables along with some constraints that can be solved easily using linear equations.

Formulation of LPP: The basic steps to Formulate a Linear Programming Problem are:

Step 1: Identify the Decision Variables

Introducing Unknowns Define the unknowns. These are usually denoted by variables such as x_1, x_2, x_3, etc.

Step 2: Formulate the Objective Function

Define the objective (maximize profit; minimize cost; etc.) as a linear function of the decision variables.

Step 3: Identify the Constraints

Identify any constraints or restrictions on the decision variables and write them as linear inequalities or equations..

Step 4: Specify the Non-Negativity Constraints

In many practical problems, they are not allowed to take negative values.

Standard Form of LPP:

bjective: Optimize (Maximize or Minimize)

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to:

- $a_{1\,1}\, x_1 + a_{1\,2}\, x_2 + \ldots + a_{1\,n}\, x_n \le$ or $=$ or $\ge b_1$
- $a_{2\,1}\, x_1 + a_{2\,2}\, x_2 + \ldots + a_{2\,n}\, x_n \le$ or $=$ or $\ge b_2$
- ...
- $a_{m\,1}\, x_1 + a_{m\,2}\, x_2 + \ldots + a_{m\,n}\, x_n \le$ or $=$ or $\ge b_m$

  And $x_1, x_2, \ldots, x_n \ge 0$

  Where:

- $x_1, x_2, \ldots, x_n$ are the decision variables
- $c_1, c_2, \ldots, c_n$ are the coefficients of the objective function
- $a_{11}, a_{12}, \ldots, a_{mn}$ are the coefficients of the constraints
- $b_1, b_2, \ldots, b_m$ are the right-hand side values of the constraints

Objective Function and Constraints

Objective Function: The objective function — the thing we want to maximize or minimize. The decision variables linear function.

If the student has two products which it could produce with respective profits of $5 and $7 per unit, and if the decision variables x_1 and x_2 define the products, the function that will define the maximized profit would be: $Z = 5x_1 + 7x_2$

Constraints: Constraint 1 : x1 + 2x2 = 0 The decision variables are the variables whose optimal values are to be determined. Equations and inequalities are expressed as linear equations or inequalities.

Types of Constraints:

1. **Resource Constraints**: These are restrictions on the available resources such as raw materials, man-hours, machine ability, etc. For example, if it takes hours to produce product 1 and hours to produce product 2 and the total amount of labor available is 60 hours then the restriction would be: $2x_1 + 3x_2 \leq 60$

2. **Market Constraints:** Constraints that arise from the demand on the market or contractual terms. For example, assume that the market demand for product 1 is at most 15 units, then: $x_1 \leq 15$

3. **Balance Constraints**: Equation that balances inputs vs outputs. Example: In a transportation problem, the amount shipped from a source must equal the available supply: $x_1 + x_2 + x_3 = 100$

4. **Non-Negativity Constraints**: Ensure that the decision variables are non-negative. Example: $x_1 \geq 0$, $x_2 \geq 0$

Canonical Form and Standard Form:

1. **Canonical Form**:

- All constraints are inequalities of the form ≤
- All variables are non-negative
- The objective function is to be maximized

2. **Standard Form**:

- All constraints are equations
- All variables are non-negative
- The objective function is to be maximized

Conversion between forms often involves adding slack variables (for ≤ constraints), surplus variables (for ≥ constraints), and artificial variables (for = constraints).

Unit 13Graphical Method of Solution

The graphical method is a visual approach to solving linear programming problems with two decision variables. It provides intuitive insights into the nature of the solution and is particularly useful for understanding the concepts of feasible region and optimal solution.

Steps in the Graphical Method:

1. **Plot the Constraints**: Graph each constraint as a line on a coordinate plane.
2. **Identify the Feasible Region**: Determine the region that satisfies all constraints.
3. **Find the Corner Points**: Identify the vertices of the feasible region.
4. **Evaluate the Objective Function**: Calculate the value of the objective function at each corner point.
5. **Determine the Optimal Solution**: Select the corner point that gives the optimal value of the objective function.

Graphical Representation of Constraints: For a constraint of the form ax + by ≤ c:-

1. Draw the line ax + by = c
2. Determine which side of the line satisfies the inequality
   - For ≤ constraints, the region below the line is feasible
   - For ≥ constraints, the region above the line is feasible

For non-negativity constraints (x ≥ 0, y ≥ 0), the feasible region is restricted to the first quadrant.

Feasible Region: Hence, this feasible solutions region is the set of all points that satisfying all the constraints of the problem. It accounts for every potential solution in the abstract sense of the problem.

Properties of the feasible region: Whenever it is always a convex polygon or unbounded convex region.

Note: if the feasible region is unbounded in the direction of optimization, then the problem might be unbounded.

Optimal Solution:

The best solution is one of the points in the feasible region with the best value of the objective function.

Important properties:

The best solution is found at one of the corners (or vertices) of the solution space.Since the optimal solution may not be unique, and for one or more of these solutions, the objective function will be parallel to a boundary of the feasible region.If there is no feasible region the solution set (value) is empty.The solution is unbounded when the feasible region is unbounded and the objective function goes to infinity (for maximization) or (for minimization) in that direction.

Linear programming problems examples

Example 1: Production Logistics

A furniture manufacturing company produces **tables** and **chairs**. The production process involves **carpentry** and **finishing**, with the following constraints:

- **Each table** requires **4 hours** of carpentry and **2 hours** of finishing.
- **Each chair** requires **3 hours** of carpentry and **1 hour** of finishing.
- The **carpentry department** has a total of **240 hours** available.
- The **finishing department** has a total of **100 hours** available.
- The profit per **table** is **$70**, and per **chair** is **$50**.

Step 1: Specify the decision variables

Let:

- x_1 = Number of **tables** produced
- x_2 = Number of **chairs** produced

Step 2: Define the objective function

The goal is to **maximize** profit:

$$Z = 70x_1 + 50x_2$$

Step 3: Understand the restrictions

1. **Carpentry constraint:**

$$4x_1 + 3x_2 \leq 240$$

2. **Finishing constraint:**

$$2x_1 + x_2 \leq 100$$

3. **Non-negativity constraints:**

$$x_1 \geq 0, \quad x_2 \geq 0$$

Step 4: Solve graphically

To graph the constraints, determine the intercepts:

1. For $4x_1 + 3x_2 = 240$:
   - When $x_1 = 0, x_2 = 80$
   - When $x_2 = 0, x_1 = 60$
2. For $2x_1 + x_2 = 100$:
   - When $x_1 = 0, x_2 = 100$
   - When $x_2 = 0, x_1 = 50$

The **feasible region** is the bounded area formed by these lines and the non-negativity constraints.

Step 5: Determine the Optimal Solution

Evaluate the **profit function** at the **corner points** of the feasible region:

- **(0, 0):**

$$Z = 70(0) + 50(0) = 0$$

- **(0, 80):**

$$Z = 70(0) + 50(80) = 4000$$

- **(30, 40):** *(Intersection of both constraints)*

$$Z = 70(30) + 50(40) = 2100 + 2000 = 4100$$

- **(50, 0):**

$$Z = 70(50) + 50(0) = 3500$$

Step 6: Conclusion

The maximum profit is **$4100**, achieved by producing **30 tables** and **40 chairs**.

Example 2: Diet Problem

A dietitian plans to prepare a meal by using two types of food: Food A and Food B. Food A contains 1 unit of protein, 2 units of carbohydrates, and 1 unit of fat per unit. The meal plan must contain at least 12 units of protein, at least 8 units of carbohydrate and at least 8 units of fat. Suppose that the cost of food A is 3.00 in dollars per unit, and the cost for food B is 4.00 dollars per unit: - How many units of each food should be included to minimize cost?

Step 1: Define decision variables

- Let x_1 = units of Food A
- Let x_2 = units of Food B

Step 2: Formulate the objective function

The goal is to **minimize** the total cost:

$$Z = 3x_1 + 4x_2$$

where **$3** is the cost per unit of **Food A** and **$4** is the cost per unit of **Food B.**

Step 3: Identify the constraints

The dietary requirements impose the following constraints:

1. **Protein Constraint:**

$$3x_1 + 2x_2 \geq 12$$

2. **Carbohydrate Constraint:**

$$2x_1 + x_2 \geq 8$$

3. **Fat Constraint:**

$$x_1 + 2x_2 \geq 8$$

4. **Non-Negativity Constraints:**

$$x_1 \geq 0, \quad x_2 \geq 0$$

Step 4: Solve graphically

Let's graph the constraints:

- For $3x_1 + 2x_2 = 12$:
  - When $x_1 = 0, x_2 = 6$
  - When $x_2 = 0, x_1 = 4$
- For $2x_1 + x_2 = 8$:
  - When $x_1 = 0, x_2 = 8$
  - When $x_2 = 0, x_1 = 4$
- For $x_1 + 2x_2 = 8$:
  - When $x_1 = 0, x_2 = 4$
  - When $x_2 = 0, x_1 = 8$

The feasible region is bounded by these lines and the non-negativity constraints.

The corner points are:

- (4, 0): $Z = 3 \times 4 = 12$
- (2, 3): $Z = 3 \times 2 + 4 \times 3 = 18$
- (0, 8): $Z = 4 \times 8 = 32$

The optimal solution is at (4, 0) with a minimum cost of $12.

Example 3: Transportation Problem

A company has two factories and three warehouses. The factories can produce 100 and 150 units per day, respectively. The warehouses require 60, 80, and 110 units per day, respectively. The transportation costs (in dollars per unit) are given in the following table:

From/To	Warehouse 1	Warehouse 2	Warehouse 3
Factory 1	10	12	8
Factory 2	15	9	11

How should the company distribute its products to minimize the total transportation cost?

Step 1: Define the decision variables

- Let x_{11} = units shipped from Factory 1 to Warehouse 1
- Let x_{12} = units shipped from Factory 1 to Warehouse 2
- Let x_{13} = units shipped from Factory 1 to Warehouse 3
- Let x_{21} = units shipped from Factory 2 to Warehouse 1
- Let x_{22} = units shipped from Factory 2 to Warehouse 2
- Let x_{23} = units shipped from Factory 2 to Warehouse 3

Step 2: Formulate the objective function

The goal is to **minimize** the total transportation cost:

$$Z = 10x_{11} + 12x_{12} + 8x_{13} + 15x_{21} + 9x_{22} + 11x_{23}$$

where each coefficient represents the **cost per unit** for transporting goods between a factory and a warehouse.

Step 3: Identify the constraints

Supply Constraints (Factory Capacity):

- **Factory 1 can supply up to 100 units:**

$$x_{11} + x_{12} + x_{13} = 100$$

- **Factory 2 can supply up to 150 units:**

$$x_{21} + x_{22} + x_{23} = 150$$

Demand Constraints (Warehouse Requirements):

- **Warehouse 1 requires 60 units:**

$$x_{11} + x_{21} = 60$$

- **Warehouse 2 requires 80 units:**

$$x_{12} + x_{22} = 80$$

- **Warehouse 3 requires 110 units:**

$$x_{13} + x_{23} = 110$$

Non-Negativity Constraints:

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0$$

Note: This problem has too many variables to solve graphically and would typically be solved using the simplex method or specialized transportation algorithms.

Unbounded Solution

An LPP has an unbounded solution when either the value of the objective function (for maximum) or the value of the objective function (min) can go on increasing or decreasing indefinitely without contradicting the constraints.An unbounded solution occurs when the feasible region goes on forever in the direction of increasing (or decreasing) objevtive function values. This occurs when the constraints on the optimization do not sufficiently bound the feasible region in the optimization direction.

Unit 14Transportation Problem

Transportation problem is a special case of linear programming problem that aims to arrive at a transportation schedule that minimizes the cost of transporting good from suppliers to consumers; the transport schedule specifies the amount of goods must be transported from each source to each destination.

We have a transportation problem where We are transporting goods from a set of supply points (sources) to a set of demand points (destinations). The supply is fixed for each source and demand is fixed for each destination. The goal is to figure out how much to ship from each source to each destination while minimizing the cost of the total transportation.The transportation problem is a special type of linear programming problem designed to determine the optimal pattern for distributing goods from multiple sources to multiple destinations, minimizing the total transportation cost.Description: In a transportation problem, goods are transported from a set of supply points (sources) to a set of demand points (destinations). Each source has a fixed supply, and each destination has a fixed demand. The objective is to determine how much to ship from each source to each destination to minimize the total transportation cost.

Key Components:

1. Sources: Points from which goods are transported (e.g., factories, warehouses)
2. Destinations: Points to which goods are transported (e.g., retail stores, distribution centers)
3. Supply: Amount of goods available at each source
4. Demand: Amount of goods required at each destination

5. Unit Transportation Cost: Cost of transporting one unit from a source to a destination

Mathematical Formulation:

- Let x_{ij} be the quantity transported from source i to destination j
- Let c_{ij} be the unit cost of transportation from source i to destination j
- Let a_i be the supply at source i
- Let b_j be the demand at destination j

Objective function: Minimize $Z = \sum\sum c_{ij} \cdot x_{ij}$

Subject to:

- $\sum x_{ij} = a_i$ for all i (supply constraints)
- $\sum x_{ij} = b_j$ for all j (demand constraints)
- $x_{ij} \geq 0$ for all i, j (non-negativity constraints)

Types of Transportation Problems:

1. Balanced Transportation Problem: Total supply equals total demand ($\sum a_i = \sum b_j$)
2. Unbalanced Transportation Problem: Total supply does not equal total demand
   - If $\sum a_i > \sum b_j$ (excess supply), introduce a dummy destination
   - If $\sum a_i < \sum b_j$ (excess demand), introduce a dummy source

Methods to Solve Transportation Problem

Several methods exist to solve transportation problems, each with its own approach and efficiency. These methods typically involve finding an initial basic feasible solution and then improving it to reach the optimal solution.

1. Initial Basic Feasible Solution Methods

a) North-West Corner Method

Description: The North-West Corner method starts at the top-left (north-west) cell of the transportation table and allocates as much as possible, considering the supply and demand constraints. It then moves to the next cell (either right or down) and continues until all supplies and demands are satisfied.

Steps:

1. Start at the top-left cell (i=1, j=1)
2. Allocate x11 = min(a1, b1)
3. Adjust supply and demand: a1 = a1 - x11, b1 = b1 - x11
4. If a1 = 0, move down to the next row (i=2, j=1)
5. If b1 = 0, move right to the next column (i=1, j=2)
6. If both a1 = 0 and b1 = 0, move diagonally (i=2, j=2)
7. Repeat until all allocations are made

Numerical Problem: Consider a transportation problem with 3 sources and 4 destinations:

Given Data:

- **Supply:**
  - Source 1: $a_1 = 100$
  - Source 2: $a_2 = 150$
  - Source 3: $a_3 = 50$
- **Demand:**
  - Destination 1: $b_1 = 75$
  - Destination 2: $b_2 = 125$
  - Destination 3: $b_3 = 50$
  - Destination 4: $b_4 = 50$

Using the North-West Corner method:

1. **Allocate to x_{11} (Source 1 → Destination 1):**
   - $x_{11} = \min(100, 75) = 75$
   - **Update:**
     - $a_1 = 100 - 75 = 25$
     - $b_1 = 75 - 75 = 0$
2. **Move right to x_{12} (Source 1 → Destination 2) since $b_1 = 0$:**
   - $x_{12} = \min(25, 125) = 25$
   - **Update:**
     - $a_1 = 25 - 25 = 0$
     - $b_2 = 125 - 25 = 100$
3. **Move down to x_{22} (Source 2 → Destination 2) since $a_1 = 0$:**
   - $x_{22} = \min(150, 100) = 100$
   - **Update:**
     - $a_2 = 150 - 100 = 50$
     - $b_2 = 100 - 100 = 0$
4. **Move right to x_{23} (Source 2 → Destination 3) since $b_2 = 0$:**
   - $x_{23} = \min(50, 50) = 50$
   - **Update:**
     - $a_2 = 50 - 50 = 0$
     - $b_3 = 50 - 50 = 0$
5. **Move down to x_{34} (Source 3 → Destination 4) since $a_2 = 0$ and $b_3 = 0$:**
   - $x_{34} = \min(50, 50) = 50$
   - **Update:**
     - $a_3 = 50 - 50 = 0$
     - $b_4 = 50 - 50 = 0$

Initial Basic Feasible Solution:

The allocated values are:

- $x_{11} = 75$
- $x_{12} = 25$
- $x_{22} = 100$
- $x_{23} = 50$
- $x_{34} = 50$

This provides the **initial feasible solution** for the transportation problem.

b) Least Cost Method

Description: The Least Cost method prioritizes allocations to cells with the lowest transportation costs. It identifies the cell with the minimum cost and allocates as much as possible, then moves to the next least-cost cell.

Steps:

1. Identify the cell with the minimum transportation cost
2. Allocate as much as possible to this cell: xij = min(ai, bj)
3. Adjust supply and demand: ai = ai - xij, bj = bj - xij
4. If either ai = 0 or bj = 0, cross out the respective row or column
5. Repeat with the next lowest-cost cell among the remaining cells
6. Continue until all allocations are made

Numerical Problem: Consider a transportation problem with 3 sources and 3 destinations, with the following cost matrix:

Given Cost Matrix:

D1	D2	D3	Supply	
S1	2	3	4	100
S2	3	2	1	150

S3	4	3	2	50
Demand	75	125	100	

Using the Least Cost method:

1. **Allocate at (S2, D3)** (Minimum cost = 1):
   - Allocate $x_{23} = \min(150, 100) = 100$
   - Update: $a_2 = 150 - 100 = 50$, $b_3 = 100 - 100 = 0$
2. **Allocate at (S1, D1)** (Next minimum cost = 2):
   - Allocate $x_{11} = \min(100, 75) = 75$
   - Update: $a_1 = 100 - 75 = 25$, $b_1 = 75 - 75 = 0$
3. **Allocate at (S2, D2)** (Next minimum cost = 2):
   - Allocate $x_{22} = \min(50, 125) = 50$
   - Update: $a_2 = 50 - 50 = 0$, $b_2 = 125 - 50 = 75$
4. **Cell (S3, D3) is skipped** as $b_3 = 0$.

5. **Allocate at (S1, D2)** (Next minimum cost = 3):
   - Allocate $x_{12} = \min(25, 75) = 25$
   - Update: $a_1 = 25 - 25 = 0$, $b_2 = 75 - 25 = 50$
6. **Allocate at (S3, D2)** (Next minimum cost = 3):
   - Allocate $x_{32} = \min(50, 50) = 50$
   - Update: $a_3 = 50 - 50 = 0$, $b_2 = 50 - 50 = 0$

Thus, the initial basic feasible solution is:

$$x_{11} = 75, \quad x_{12} = 25, \quad x_{22} = 50, \quad x_{23} = 100, \quad x_{32} = 50$$

c) Vogel's Approximation Method (VAM)

Description: Vogel's Approximation Method is more sophisticated and generally produces better initial solutions. It calculates penalty costs for each row and column as the difference between the two lowest costs in that row or column. It then makes allocations to minimize these penalties.

Steps:

1. Calculate the penalty for each row and column:

- Row penalty = difference between the two lowest costs in the row
- Column penalty = difference between the two lowest costs in the column

2. Identify the row or column with the highest penalty
3. In that row or column, find the cell with the minimum cost
4. Allocate as much as possible to this cell: xij = min(ai, bj)
5. Adjust supply and demand: ai = ai - xij, bj = bj - xij
6. If either ai = 0 or bj = 0, cross out the respective row or column
7. Recalculate penalties and repeat until all allocations are made

Numerical Problem: Consider a transportation problem with 3 sources and 3 destinations, with the following cost matrix:

Step 1: Given Cost Matrix and Supply-Demand Data

	D1	**D2**	**D3**	**Supply**
S1	2	3	4	100
S2	3	2	1	150
S3	4	3	2	50
Demand	75	125	100	

Using Vogel's Approximation Method:

1. Calculate penalties:

Step 2: Compute Initial Penalties

- Row Penalties:
  - Row 1: $3 - 2 = 1$
  - Row 2: $3 - 1 = 2$
  - Row 3: $4 - 2 = 2$
- Column Penalties:
  - Column 1: $3 - 2 = 1$
  - Column 2: $3 - 2 = 1$
  - Column 3: $4 - 1 = 3$

Step 3: First Allocation

- Highest penalty: 3 (Column 3)
- Lowest cost in Column 3: 1 (Cell S2, D3)
- Allocation:

$$x_{23} = \min(150, 100) = 100$$

- Update Supply & Demand:
  - $a_2 = 150 - 100 = 50$
  - $b_3 = 100 - 100 = 0$

Since $b_3 = 0$, Column 3 is **satisfied and removed from further**

Step 4: Recalculate Penalties

Since **Column 3 is eliminated**, we now recalculate penalties for the remaining rows and columns:

- **Updated Cost Matrix (without Column 3):**

	D1	D2	Supply
S1	2	3	100
S2	3	2	50
S3	4	3	50
Demand	75	125	

- New Row Penalties:
  - Row 1: $3 - 2 = 1$
  - Row 2: $3 - 2 = 1$
  - Row 3: $4 - 3 = 1$
- New Column Penalties:
  - Column 1: $3 - 2 = 1$
  - Column 2: $3 - 2 = 1$

Now, we continue with the **next highest penalty allocation** in the subsequent steps.

a) Stepping Stone Method

Description: The Stepping Stone method is used to check if the current basic feasible solution is optimal and, if not, to improve it. It evaluates the cost change when introducing a new variable into the basis.

Steps:

1. For each non-basic (empty) cell, create a closed path that:
   - Starts and ends at the empty cell
   - Only turns at allocated cells
   - Consists of horizontal and vertical segments
2. Calculate the net cost change for each empty cell:
   - Add the cost when moving in the "+" direction
   - Subtract the cost when moving in the "-" direction
3. If all net cost changes are non-negative, the current solution is optimal
4. Otherwise, select the cell with the most negative net cost change
5. Allocate as much as possible to this cell by:
   - Finding the minimum allocation among the "-" cells in the path
   - Adding this amount to the "+" cells in the path
   - Subtracting this amount from the "-" cells in the path
6. Repeat until all net cost changes are non-negative

Numerical Problem: Consider an initial basic feasible solution with the following allocations:

	D1	D2	D3
S1	75	25	0
S2	0	50	100
S3	0	50	0

With the cost matrix:

	D1	D2	D3
S1	2	3	4
S2	3	2	1
S3	4	3	2

Evaluating the empty cell (2,1):

- Path: (2,1) → (2,2) → (1,2) → (1,1) → (2,1)
- Net cost change: c21 - c22 + c12 - c11 = 3 - 2 + 3 - 2 = 2

Since the net cost change is positive, this cell should remain empty.

Continue this process for all empty cells to determine if the solution is optimal.

b) Modified Distribution Method (MODI)

Description: The Modified Distribution Method is a more efficient approach to check for optimality. It uses dual variables to calculate the net cost changes without explicitly constructing closed paths.

Steps:

1. Assign dual variables ui and vj to each row and column
2. For each allocated cell (i,j), set ui + vj = cij
3. Solve the system of equations for ui and vj (set u1 = 0 for convenience)
4. For each empty cell (i,j), calculate the net cost change: dij = cij - ui - vj
5. If all dij ≥ 0, the current solution is optimal
6. Otherwise, select the cell with the most negative dij
7. Construct a closed path and make the improvement as in the Stepping Stone method
8. Repeat until all dij ≥ 0

Numerical Problem: Using the same example as before:

	D1	D2	D3
S1	75	25	0
S2	0	50	100
S3	0	50	0

With the cost matrix:

	D1	D2	D3
S1	2	3	4
S2	3	2	1
S3	4	3	2

Set $u_1 = 0$ and solve for the dual variables:

- For cell (1,1): $u_1 + v_1 = c_{11} \rightarrow 0 + v_1 = 2 \rightarrow v_1 = 2$
- For cell (1,2): $u_1 + v_2 = c_{12} \rightarrow 0 + v_2 = 3 \rightarrow v_2 = 3$
- For cell (2,2): $u_2 + v_2 = c_{22} \rightarrow u_2 + 3 = 2 \rightarrow u_2 = -1$
- For cell (2,3): $u_2 + v_3 = c_{23} \rightarrow -1 + v_3 = 1 \rightarrow v_3 = 2$
- For cell (3,2): $u_3 + v_2 = c_{32} \rightarrow u_3 + 3 = 3 \rightarrow u_3 = 0$

Calculate d_{ij} for empty cells:

- $d_{13} = c_{13} - u_1 - v_3 = 4 - 0 - 2 = 2$
- $d_{21} = c_{21} - u_2 - v_1 = 3 - (-1) - 2 = 2$
- $d_{31} = c_{31} - u_3 - v_1 = 4 - 0 - 2 = 2$
- $d_{33} = c_{33} - u_3 - v_3 = 2 - 0 - 2 = 0$

Since all $d_{ij} \geq 0$, the current solution is optimal.

Practical Applications: Real World Applications of Transportation Problems Transport problems have a lot of real-world applications in several industries and domains. These applications illustrate the diverse range and significance of transportation modeling approaches.

Supply Chain Management: Supply Chain Management: Supply chain management refers to the process of overseeing and managing the flow of goods and services from suppliers to manufacturers, distributors, retailers, and consumers. Transportation difficulties serve to enhance the flow of goods through this network.

Applications:

• Defining how much to ship from factories to warehouses

• Reallocating goods from distribution centers to bricks-and-mortar stores

• Creating the logistics for bringing raw materials from suppliers to the manufacturing plants

• Bringing optimization of inventory levels through supply chain

For example, one company has 3 manufacturing plants and needs to supply 5 regional warehouses. The production capacities of the plants are different from each other, as well as the demand requirements of the warehouses. These shipping fees depend on the distance travelled as well as the method of shipping. Formulating and solving a transportation problem would allow the company to minimize the overall transportation costs from the plants to the warehouse and satisfy all demand.

Numerical problem: A furniture company has 3 factories with monthly production capacity of 1000, 1500 and 800 units respectively. It has to deliver 4 warehouses with monthly demands of 900, 1200, 700, and 500 units. Transport costs (in $ per unit) are:

	W1	W2	W3	W4
Factory 1	10	12	9	11
Factory 2	8	7	10	12
Factory 3	14	9	16	13

Formulate and solve this transportation problem to determine the optimal shipment plan.

Solution:

Step 1: Verify Balance Condition

Total supply:

$$1000 + 1500 + 800 = 3300$$

Total demand:

$$900 + 1200 + 700 + 500 = 3300$$

Since total supply equals total demand, this is a **balanced transportation problem.**

Step 2: Initial Solution Using Least Cost Method

1. **Minimum cost = 7 at (S2, D2)**
   - Allocate $x_{22} = \min(1500, 1200) = 1200$
   - Update: $a_2 = 1500 - 1200 = 300$, $b_2 = 1200 - 1200 = 0$
2. **Next minimum cost = 8 at (S2, D1)**
   - Allocate $x_{21} = \min(300, 900) = 300$
   - Update: $a_2 = 300 - 300 = 0$, $b_1 = 900 - 300 = 600$
3. **Next minimum cost = 9 at (S1, D3) and (S3, D2)**
   - For $(S1, D3)$: Allocate $x_{13} = \min(1000, 700) = 700$
   - Update: $a_1 = 1000 - 700 = 300$, $b_3 = 700 - 700 = 0$
   - No allocation at $(S3, D2)$ since $b_2 = 0$.

4. Next minimum cost = 10 at (S1, D1)
   - Allocate $x_{11} = \min(300, 600) = 300$
   - Update: $a_1 = 300 - 300 = 0$, $b_1 = 600 - 300 = 300$
5. Next minimum cost = 11 at (S1, D4)
   - No allocation possible since $a_1 = 0$.
6. Next minimum cost = 13 at (S3, D4)
   - Allocate $x_{34} = \min(800, 500) = 500$
   - Update: $a_3 = 800 - 500 = 300$, $b_4 = 500 - 500 = 0$
7. Next minimum cost = 14 at (S3, D1)
   - Allocate $x_{31} = \min(300, 300) = 300$
   - Update: $a_3 = 300 - 300 = 0$, $b_1 = 300 - 300 = 0$

Step 3: Initial Basic Feasible Solution

	D1	D2	D3	D4	Supply
S1	**300**	**0**	**700**	**0**	**1000**
S2	**300**	**1200**	**0**	**0**	**1500**
S3	**300**	**0**	**0**	**500**	**800**
Demand	**900**	**1200**	**700**	**500**	

Total Initial Cost Calculation:

$$10(300) + 9(700) + 8(300) + 7(1200) + 14(300) + 13(500)$$

$$= 3000 + 6300 + 2400 + 8400 + 4200 + 6500 = 30,800$$

Step 4: Optimal Solution Using MODI Method

	D1	D2	D3	D4	Supply
S1	300	0	700	0	1000
S2	600	900	0	0	1500
S3	0	300	0	500	800
Demand	900	1200	700	500	

Total Optimal Cost Calculation:

$$10(300) + 9(700) + 8(600) + 7(900) + 14(0) + 13(500)$$

$$= 3000 + 6300 + 4800 + 6300 + 0 + 6500 = 29,900$$

Thus, the **optimal solution** has a **total transportation cost of $29,900.**

Distribution Network Design: This paper primarily focuses on the design of distribution networks, which involves optimizing the allocation of distribution centers and the allocation of flow of goods through this network. Optimization of Distribution Strategy Aided by Transportation Problems.

Applications:

- Locating warehouses to minimize transportation costs
- Determining the optimal number and size of distribution centers
- Allocating service territories to distribution centers
- Planning cross-docking operations

Example: A retail chain is planning its distribution network and needs to decide which of 5 potential warehouse locations

Multiple Choice Questions (MCQs)

1. The objective of a Linear Programming Problem is to:
   a) Maximize or minimize a linear function
   b) Solve nonlinear equations

2. c) Use graphical charts only

   d) Eliminate all variables

3. A feasible region in LPP is defined as:

   a) The area containing no solution

   b) The set of all possible solutions that satisfy the constraints

   c) The maximum value of the objective function

   d) The region where no constraints apply

4. In a Linear Programming Problem, constraints are represented by:

   a) Linear equations

   b) Quadratic equations

   c) Exponential equations

   d) Logarithmic equations

5. If a Linear Programming Problem has no feasible solution, it is called:

   a) Redundant

   b) Infeasible

   c) Unbounded

   d) Multiple solution case

6. When a Linear Programming Problem has an unbounded solution, it means:

   a) No optimal solution exists

   b) The solution can be infinitely large

   c) The problem is incorrect

   d) The constraints are not satisfied

7. In the graphical method of solving LPP, the optimal solution lies:

   a) Inside the feasible region

   b) Outside the feasible region

   c) At the origin

   d) At one of the corner points of the feasible region

8. The transportation problem is used to:

   a) Optimize the distribution of goods

   b) Solve LPP using graphical methods

c) Solve nonlinear programming problems

d) Increase transportation costs

9. The basic feasible solution in a transportation problem is determined using:

   a) Simplex method

   b) North-West Corner Rule

   c) Gauss-Jordan method

   d) Newton-Raphson method

10. The transportation problem deals with:

    a) Assigning tasks to workers

    b) Allocating resources efficiently

    c) Finding the least cost of transporting goods

    d) Maximizing the distance traveled

11. A redundant constraint in LPP is:

    a) A constraint that does not affect the feasible region

    b) A constraint that eliminates the optimal solution

    c) A constraint that adds more variables

    d) A constraint that changes the objective function

Short Answer Questions

1. Define Linear Programming Problem and explain its components.
2. What is the significance of the feasible region in LPP?
3. Explain the graphical method for solving LPP with an example.
4. What are the different types of solutions in LPP? Explain.
5. Describe the transportation problem and its real-world applications.

Module V THEORY OF INDICES AND LOGARITHMS

Structure

OBJECTIVES

To understand the concept of indices and their rules.

To explore the properties and applications of logarithms.

To apply indices and logarithms in business and financial calculations

Unit 15Theory of Indices

The concept of indices, also called exponents or powers, allows us to write repeated multiplication of a number with itself in a shorter form. 1 In the expression an, we call a the base and n the index (or exponent). 2 It means 'a' raised to the power of 'n'

Example:

- 23=2×2×2=8
- 2 is the base and 3 is the index here.
- 54=5×5×5×5=625
- In this case, 5 is the base and, 4 is the index.

From mathematical perspective, indices as taught in school are unique to utilise but with its knowledge you need to stay uptight with your subjects as it gives you the distance of power and ensures a sizeable representation for large or small numbers. 3

Laws of Indices are a set of rules to simplify power operations. 4 These laws are useful when working with expressions with indices.

1. Product Law:

$$a^m \times a^n = a^{m+n}$$

When multiplying two powers with the same base, add the exponents

Description: This law simplifies the multiplication of expressions with the same base.

Examples:

- $2^3 \times 2^4 = 2^{3+4} = 2^7 = 128$
- $x^2 \times x^5 = x^{2+5} = x^7$
- $3^2 \times 3^1 = 3^{2+1} = 3^3 = 27$

Numerical Problem

1. Calculate $7^3 \times 7^5$.
   - Solution: $7^3 \times 7^5 = 7^{3+5} = 7^8 = 5,764,801$.
2. Calculate $10^2 \times 10^3$.
   - Solution: $10^2 \times 10^3 = 10^{2+3} = 10^5 = 100,000$.

2. Quotient Law:

$$\frac{a^m}{a^n} = a^{m-n}$$

For the same base of powers being divided, the exponents are subtracted. 8

Description: According to this law, the division of expressions with the same base is simplified.

3. Power Law:

$$(a^m)^n = a^{m \times n}$$

When you have a power raised to another power, then you get to multiply the exponents.

Description: This law helps us in simplifying the exponentiation of power.

Examples:

- $(3^2)^3 = 3^{2\times3} = 3^6 = 729$
- $(z^4)^5 = z^{4\times5} = z^{20}$
- $(4^3)^2 = 4^{3\times2} = 4^6 = 4,096$

Numerical Problem

1. Calculate $(4^3)^4$.
   - Solution: $(4^3)^4 = 4^{3\times4} = 4^{12} = 16,777,216.$
2. Calculate $(2^5)^3$.
   - Solution: $(2^5)^3 = 2^{5\times3} = 2^{15} = 32,768.$

2. Zero Exponent Law:

$$a^0 = 1 \text{ (where } a \neq 0\text{)}$$

A number (any number) raised to the power of 0 is equal to 1.

Description: This law states that the value of a base raised to the power of zero.

Examples:

- 10^0=1
- $(-7)^0$=1
- x^0=111

Numerical Problem

1. Calculate 15^0.
   - Solution: $15^0 = 1$.
2. Calculate $(-23)^0$.
   - Solution: $(-23)^0 = 1$.

5. Power of a Product Law:

$$(ab)^n = a^n \times b^n$$

That is, the power of the product equals the product of the powers.

Description: This rule applies the exponent to the factors of a product.

Examples:

- $(2 \times 3)^4 = 2^4 \times 3^4 = 16 \times 81 = 1,296$
- $(xy)^3 = x^3y^3$
- $(5 \times 2)^2 = 5^2 \times 2^2 = 25 \times 4 = 100$

Numerical Problem

1. Calculate $(3 \times 4)^3$.
   - Solution: $(3 \times 4)^3 = 3^3 \times 4^3 = 27 \times 64 = 1,728$.
2. Calculate $(7 \times 2)^2$.
   - Solution: $(7 \times 2)^2 = 7^2 \times 2^2 = 49 \times 4 = 196$.

6. Power of a Quotient Law:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \text{ (where } b \neq 0\text{)}$$

This is also true when the operations are turned around, with emphasis on being quotated:

Description: Exponential Rule- This law distributes the exponent across the numerator and denominator of a fractional power.

Examples:

- $\left(\frac{5}{2}\right)^3 = \frac{5^3}{2^3} = \frac{125}{8}$
- $\left(\frac{x}{y}\right)^4 = \frac{x^4}{y^4}$
- $\left(\frac{6}{3}\right)^2 = \frac{6^2}{3^2} = \frac{36}{9} = 4$

Numerical Problem

1. Calculate $\left(\frac{4}{3}\right)^2$.
   - Solution: $\left(\frac{4}{3}\right)^2 = \frac{4^2}{3^2} = \frac{16}{9}$.
2. Calculate $\left(\frac{10}{5}\right)^3$.
   - Solution: $\left(\frac{10}{5}\right)^3 = \frac{10^3}{5^3} = \frac{1000}{125} = 8$.

Indices: Positive, Negative, and Fractional Indices

Therefore, indices can be positive, negative and decimal, all with their own meanings.

1. Positive Indices

As we covered previously, positive indices indicate repeated multiplication..

- Example: $4^3=4\times4\times4=64$

2. Negative Indices:

$$a^{\ n} = \frac{1}{a^n} \text{ (where } a \neq 0)$$

Its base will be raised to a positive index. A negative index shows the reciprocal. 13

Description: Negative indices are used to show reciprocals and inverse relationships. 14

Examples:

- $2^{\ 3} = \frac{1}{2^3} = \frac{1}{8}$.
- $10^{\ 2} = \frac{1}{10^2} = \frac{1}{100} = 0.01$.
- $5^{\ 1} = \frac{1}{5}$.

Unit 16 Logarithms: An Introduction to Logarithms

At their heart, logarithms are about inverting the operation of exponentiation. 1 Not “What is 2 to the 3rd power?” (Which is 8), logarithms ask us, “Which power do we need to raise 2 to to equal 8?” In this case, the answer is 3.Understanding the Meaning & Concept of Logarithms In a formal sense if an equation takes the following format:

$b^y = x$

Where:

- b is the base (a positive number, not equal to 1)
- y is the exponent3
- x is the result

The logarithm of x (a positive number), with base b (also a positive number) is then defined in the following way:

$$\log_b (x) = y$$

It is technically the (m) exponent or power to which the base (b) must be raised to be equal to the number (x). 4

Conceptual Understanding

Imagine a number line. We walked along this line when we multiplied numbers. Exponentiation is one manner of leaping forward along this line. Logarithms, in other words, are a way to measure “how far” we’ve progressed.

Why Logarithms Are Useful

- Logarithms convert multiplication to addition and division to extraction, making them very useful for simplifying complex calculations. 8
- Scaling Data: They are great for displaying data that ranges over multiple orders of magnitude like in the case of scientific measurements and financial data. 9

- Solving Exponential Equations: In equations where the unknown variable is an exponent, logarithms are used to resolve them. 10

Decibels: Sound is measured in decibels, which is a logarithmic measure of sound intensity. 11

- Chemistry: pH is measured on a log scale. 12

Laws of Logarithms

These laws provide the principles by which logarithms work, which are important for manipulating and simplifying logarithmic expressions.[13]

1. **Product Rule:**

$$\log_b(mn) = \log_b(m) + \log_b(n)$$ [14]

- The logarithm of a product is the sum of the logarithms.[15]
- Example: log

$$\log_2(8 \times 4) = \log_2(8) + \log_2(4) = 3 + 2 = 5$$

2. **Quotient Rule:**

$$\log_b\left(\frac{m}{n}\right) = \log_b(m) - \log_b(n)$$

- The logarithm of a quotient is the difference of the logarithms.[17]
- Example:

$$\text{Log}_3(81/9) = \log_3(81) - \log_3(9) = 4 - 2 = 2$$

3. **Power Rule:**

$$\log_b(m^n) = n \cdot \log_b(m)$$

- The logarithm of a number raised to a power is the power times the logarithm of the number.[18]
- Example:

$$\log_5(25^3) = 3 \times \log_5(25) = 3 \times 2 = 6$$

4. **Change of Base Rule:**

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

- This allows us to convert logarithms from one base to another.
- Example:

$$\log_2(10) = \frac{\log_{10}(10)}{\log_{10}(2)}$$

5. Logarithm of 1:

$$\log_b(1) = 0$$

- Any number raised to the power of 0 is 1.

6. Logarithm of the Base:

$$\log_b(b) = 1$$

- Any number raised to the power of 1 is itself.

Relationship between Indices (Exponents) and Logarithms

As mentioned, logarithms are the inverse of exponentiation This means:

If $b^y = x$, then $\log_b(x) = y$.

- Conversely,

$\log_b(x) = y$, then $b^y = x$.

This relationship is crucial for solving equations and converting between exponential and logarithmic forms.

Common Logarithms and Natural Logarithms: Two logarithmic bases are particularly important:

1. Common Logarithms (Base 10):

- Written as log(x) or $\log_{10}(x)$.
- Used extensively in science and engineering.
- Example: log(1000) = 3.

2. Natural Logarithms (Base e):

- Written as ln(x) or $\log_e(x)$.
- e is Euler's number (approximately 2.71828).

- Used extensively in calculus and higher mathematics.
- Example: ln(e) = 1.

Numerical Problems and Examples

Let's work through some numerical examples to solidify understanding:

1. Using Logarithm Laws

Properties and Applications

Both logarithms are basic mathematical concepts that give a means of expressing exponential relationships in reverse. 1 The logarithm of a number, broadly speaking, is the power to which a base must be raised to achieve that number.

Description:

- Logarithm is the inverse of exponentiation. 2 If $b^y=x$,

$$\log_b (x) = y.$$

- In this structure, we have the base (b), argument (x), and logarithm (y).
- Common bases include:

Base 10 (Common Logarithm): log(x) or $\log_{10}(x)$

Base e (Natural Logarithm): ln (x) or $\log_e(x)$

Examples:

- $10^2=100$, so $\log_{10}(100)=2$
- $2^3=8$, so $\log_2(8)=3$.
- $e^0=1$, so $\ln(1)=0$

Logarithm Pak Basic Properties: The knowledge of the properties of the logarithm is important because they will help us do logarithmic operations and solve equations.

1. Product Rule:

- $\log_b(mn)=\log_b(m)+\log_b(n)7$
- The logarithm of a product is equal to the sum of the logarithms.

Example:

$$\log^{10}(10\times100)=\log_{10}(10)+\log_{10}(100)=1+2=3.$$

2. Quotient Rule:

- $\log_b\left(\frac{m}{n}\right)=\log_b(m)-\log_b(n)$
- The log of a quotient is the difference of the logs.

Example:

$$\log_2\left(\frac{16}{2}\right)=\log_2(16)-\log_2(2)=4-1=3$$

3. Power Rule:

- $\log_b(m^n)=n\cdot\log_b(m)$

Example:

$$\log_{10}(10^3)=3\cdot\log_{10}(10)=3\cdot1=3.$$

4. Change of Base Rule:

- $\log_b(x)=\frac{\log_a(x)}{\log_a(b)}$
- You can change the base of log with this.

Example:

$$\log_2(5)=\frac{\log_{10}(5)}{\log_{10}(2)}\approx\frac{0.69897}{0.30103}\approx2.3219$$

5. Logarithm of 1:

- $\mathbf{\log_b(1)=0}$

- Any power 0 equals 1 (with exceptions)

:xample:

$$\log_{10}(1)=011$$

$$\ln(1)=0.12$$

6. Logarithm of the Base:

$$\log_b(b)=1$$

- Any number with an exponent of 1 is itself. 13

Logarithmic Calculations

Logarithmic calculations are the ones in which logarithmic properties are applied to solve problems or to simplify complicated expression. 14

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1. Simplify $\log_3(9 \times 27)$:

$$\log_3(9) + \log_3(27) = \log_3(3^2) + \log_3(3^3) = 2 + 3 = 5$$

2. Simplify $\log_{10}(1000/10)$:

$$\log_{10}(1000) - \log_{10}(10) = 3 - 1 = 2$$

3. Evaluate $\log_2(45)$ (Incorrect formulation in original text)

4. Find $\log_{10}(8)$ given $\log_{10}(2) \approx 0.3010$:

$$\log_{10}(8) = \log_{10}(2^3) = 3 \cdot \log_{10}(2) = 3 \times 0.3010 = 0.9030$$

5. Find $\log_3(25)$ given $\log_3(5) = 1.465$:

$$\log_3(25) = \log_3(5^2) = 2 \cdot \log_3(5) = 2 \times 1.465 = 2.93$$

Use of Logarithm Tables

Before the advent of calculators, logarithm tables were the primary tool for performing complex calculations.

Description:

- Logarithm tables provide pre-calculated logarithms of numbers to a certain base (usually base 10).
- They consist of a characteristic (the integer part) and a mantissa (the decimal part) of the logarithm.

- The characteristic indicates the power of 10, and the mantissa represents the logarithm of the significant digits.

Using Logarithm Tables:

Finding the Logarithm of a Number:

1. Determine the characteristic based on the number's magnitude.
2. Locate the mantissa in the logarithm table using the significant digits of the number.
3. Combine the characteristic and mantissa.

Finding the Antilogarithm of a Number:

- Separate the characteristic and mantissa.
- Find the number in the antilogarithm table corresponding to the mantissa.

Adjust the decimal point based on the characteristic.

Example: Find log10(345)

Characteristic: 102≤345<103, so the characteristic is 2.

- Mantissa: Locate 34 in the leftmost column and 5 in the top row. The mantissa is approximately 0.5378.
- $\log_{10}(345) \approx 2.5378$.

Find the antilogarithm of 3.6021.

- Characteristic: 3, Mantissa: 0.6021.
- Antilogarithm: The antilogarithm of 0.6021 is approximately 4.00.
- Antilogarithm : $4.00 \times 10^3 = 4000$.

Numerical Problems using Log Tables:

- Find $\log_{10}(567)$ using log table. (approximately: 2.7536)
- Find the antilog of 1.8808 (Approx 76) using log table
- Calculate 321*4.2 using Log Table. (approximately 1348.2)
- Find 523 / 82 using Log Table (approximately 6.37)

- Calculate $231^{.5}$ (approximately 110.16) by means of Table of logs5. Solving Exponential Equations ogarithms are essential for solving exponential equations, where the ariable is in the exponent.

Description:

- Exponential equations involve an unknown variable as an exponent.[20]
- To solve them, take the logarithm of both sides and use logarithmic properties.

Description:

- Logarithmic equations hold logarithms of expressions in the unknown variable.
- Simplify using logarithmic properties and solve for the variable.

Applications of Logarithm in Real Life

The concept of logarithm has applications in several fields, helping to understand and model problems with exponential growth or decay..

Sound Intensity (Decibels):

- Decibels (dB) measure sound intensity logarithmically.
- Formula:

$$dB = 10\log_{10}\left(\frac{I}{I_0}\right)$$

where I is the sound intensity and I_0 is a reference intensity.

- Example: If the sound intensity increases by a factor of 100, the decibel level increases by:

$$10\log_{10}(100) = 10 \times 2 = 20\,dB.$$

.

Magnitude (Richter scale) of earthquake:

- The Richter scale is logarithmic in its measurement of earthquake magnitude.
- Formula:

$$M = \log_{10}\left(\frac{A}{A_0}\right)$$

where A is the amplitude of seismic waves and A_0 is a reference amplitude.

- Example: A magnitude 6 earthquake is **10 times** stronger than a magnitude 5 earthquake.

pH Scale (Acidity):

- The pH scale measures the acidity or basicity of a solution logarithmically.
- Formula:

$$pH = -\log_{10}[H^+]$$

where $[H^+]$ is the concentration of hydrogen ions.

- Example: A solution with a pH of 3 has **10 times** more hydrogen ions than a solution with a pH of 4.

Compound Interest:

- Logarithms calculate the time it takes for an investment to reach a certain amount, according to compound interest.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where:

- A is the future value,
- P is the principal,
- r is the interest rate,
- n is the number of times interest is compounded per year,
- t is the time in years.

Example: You invest $1000 at a 5% interest rate compounded annually – how long does it take to double your investment?

- $2000=1000(1+0.05)^{t}$.
- $2=(1.05)^{t}$.
- Taking the logarithm:

$$\log(2) = t\log(1.05)$$

$$t = \frac{\log(2)}{\log(1.05)}$$

$$t \approx \frac{0.3010}{0.0212} \approx 14.2 \text{ years.}$$

Radioactive Decay:

- Logarithms model radioactive decay over time.
- Formula:

$$N(t) = N_0 e^{-kt}$$

where:

  - $N(t)$ is the amount remaining at time t.
  - N_0 is the initial amount.
  - k is the decay constant.
- Example: The half-life of Carbon-14 is **5730 years**. How long does it take for **75% of the carbon-14** to decay?

$$0.25N_0 = N_0 e^{-kt}$$

$$0.25 = e^{-kt}$$

Multiple Choice Questions (MCQs)

1. The expression $a^m \times a^n$ is simplified as:

   a) a^{m+n}

   b) a^{m-n}

   c) $a^{m\times n}$

   d) $a^{m/n}$

2. The value of a^0 (where $a\neq 0$) is:

   a) 0

   b) 1

c) a

d) Undefined

3. Which of the following is NOT a law of indices?

   a) $(a^m)^n = a^{m\ n}$

   b) $a^m \times a^n = a^{m\ n}$

   c) $a^{\ m} = \frac{1}{a^m}$

   d) $\frac{a^m}{a^n} = a^{m\ n}$

4. If log101000=x\log_{10} 1000 = xlog101000=x, then the value of xxx is:

   a) 1

   b) 2

   c) 3

   d) 4

5. The logarithm of 1 to any base is always:

   a) 0

   b) 1

   c) The base itself

   d) Undefined

6. What is the logarithm of a product $\log_b(mn)$ equal to?

   a) $\log_b m + \log_b n$

   b) $\log_b m - \log_b n$

   c) $\log_b m \times \log_b n$

   d) $\frac{\log_b m}{\log_b n}$

7. The logarithm of a number is the inverse operation of:

   a) Addition

   b) Multiplication

   c) Exponentiation

   d) Division

8. The value of $\log_{10}10^5$ is:

   a) 1

   b) 5

   c) 10

   d) 50

9. Which logarithm base is commonly used in scientific calculations?

   a) Base 2

b) Base 5

c) Base 10

d) Base e

10. The natural logarithm is denoted as:

a) $\log_e$

b) $\log_2$

c) $\log_5$

d) $\log_{10}$

Short Answer Questions

1. Define indices and state the laws of indices.
2. What is the difference between common logarithms and natural logarithms?
3. Explain the relationship between indices and logarithms with an example.
4. How can logarithms be used to simplify multiplication and division?
5. Solve: log 232 = x what is the value of x?

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